

NAÏVE LIFTINGS OF DG MODULES

SAEED NASSEH, MAIKO ONO, AND YUJI YOSHINO

ABSTRACT. Let n be a positive integer, and let A be a strongly commutative differential graded (DG) algebra over a commutative ring R . Assume that

- (a) $B = A[X_1, \dots, X_n]$ is a polynomial extension of A , where X_1, \dots, X_n are variables of positive degrees; or
- (b) A is a divided power DG R -algebra and $B = A\langle X_1, \dots, X_n \rangle$ is a free extension of A obtained by adjunction of variables X_1, \dots, X_n of positive degrees.

In this paper, we study naïve liftability of DG modules along the natural injection $A \rightarrow B$ using the notions of diagonal ideals and homotopy limits. We prove that if N is a bounded below semifree DG B -module such that $\text{Ext}_B^i(N, N) = 0$ for all $i \geq 1$, then N is naïvely liftable to A . This implies that N is a direct summand of a DG B -module that is liftable to A . Also, the relation between naïve liftability of DG modules and the Auslander-Reiten Conjecture has been described.

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1. INTRODUCTION

Throughout the paper, R is a commutative ring.

Let I be an ideal of R , and assume in this paragraph that R is I -adically complete and local. When I is generated by an R -regular sequence, lifting property of finitely generated modules and of bounded below complexes of finitely generated free modules along the natural surjection $R \rightarrow R/I$ was studied by Auslander, Ding, and Solberg [2] and Yoshino [35]. Nasseh and Sather-Wagstaff [22] generalized these results to the case where I is not necessarily generated by an R -regular

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sequence. In this case, they considered the lifting property of differential graded (DG) modules along the natural map from R to the Koszul complex on a set of generators of the ideal I .

Let $A \rightarrow B$ be a homomorphism of DG R -algebras. A right DG B -module N is *liftable* to A if there is a right DG A -module M such that $N \cong M \otimes_A^{\mathbf{L}} B$ (or $N \cong M \otimes_A B$, if M and N are semifree) in the derived category $\mathcal{D}(B)$. In their recent works, Nasseh and Yoshino [27] and Ono and Yoshino [28] proved the following results on liftability of DG modules; see 2.4 and 2.8 for notation.

Theorem 1.1 ([27, 28]). *Let A be a DG R -algebra and $B = A\langle X \rangle$ be a simple free extension of A obtained by adjunction of a variable X of degree $|X| > 0$ to kill a cycle in A . Assume that N is a semifree DG B -module with $\mathrm{Ext}_B^{|X|+1}(N, N) = 0$.*

- (a) *If $|X|$ is odd, then $N \oplus N(-|X|)$ is liftable to A (that is, N is weakly liftable to A in the sense of [21, Definition 5.1]).*
- (b) *If $|X|$ is even and N is bounded below, then N is liftable to A .*

Naïve lifting property of DG modules along simple free extensions of DG algebras was introduced in [21] to obtain a new characterization of (weak) liftability of DG modules along such extensions; see [21, Theorem 6.8]. However, our study of naïve lifting property of DG modules in this paper is mainly motivated by a conjecture of Auslander and Reiten as we explain in Section 7; see Theorem 7.1. For the general definition of naïve liftability, let $A \rightarrow B$ be a homomorphism of DG R -algebras such that the underlying graded A -module B is free. Let N be a semifree right DG B -module, and denote by $N|_A$ the DG B -module N regarded as a right DG A -module via $A \rightarrow B$. We say that N is *naïvely liftable* to A if the DG B -module epimorphism $\pi_N: N|_A \otimes_A B \rightarrow N$ defined by $\pi_N(x \otimes b) = xb$ splits; see 5.1 for more details. The purpose of this paper is to prove the following result that deals with this version of liftability along finite free and polynomial extensions of DG algebras; see 2.2 and 2.5 for the definitions and notation.

Main Theorem. *Let n be a positive integer. We consider the following two cases:*

- (a) *$B = A[X_1, \dots, X_n]$ is a polynomial extension of A , where X_1, \dots, X_n are variables of positive degrees; or*
- (b) *A is a divided power DG R -algebra and $B = A\langle X_1, \dots, X_n \rangle$ is a free extension of A obtained by adjunction of variables X_1, \dots, X_n of positive degrees.*

In either case, if N is a bounded below semifree DG B -module with $\mathrm{Ext}_B^i(N, N) = 0$ for all $i > 0$, then N is naïvely liftable to A . Moreover, N is a direct sum of a DG B -module that is liftable to A .

A unified method to prove parts (a) and (b) of Theorem 1.1 is introduced in [21] using the notion of j -operators. However, as is noted in [21, 3.10], this notion cannot be generalized (in a way that useful properties of j -operators are preserved) to the case where we have more than one variable. Our approach in this paper in order to prove Main Theorem is as follows. In Section 3, we define the notions of diagonal ideals and DG smoothness, which is a generalization of the notion of smooth algebras in commutative ring theory. Then using the notion of homotopy limits, discussed in Section 4, we prove the following result in Section 5.

Theorem 1.2. *Let $A \rightarrow B$ be a DG smooth homomorphism. If N is a bounded below semifree DG B -module with $\mathrm{Ext}_B^i(N, N) = 0$ for all $i \geq 1$, then N is naïvely liftable to A . Moreover, N is a direct sum of a DG B -module that is liftable to A .*

The proof of Main Theorem then follows after we show that under the assumptions of Main Theorem, $A \rightarrow B$ is DG smooth. This takes up the entire Section 6.

2. TERMINOLOGY AND PRELIMINARIES

We assume that the reader is fairly familiar with complexes, DG algebras, DG modules, and their properties. Some of the references on these subjects are [4, 7, 12, 13]. In this section, we specify the terminology and include some preliminaries that will be used in the subsequent sections.

2.1. Throughout the paper, A is a *strongly commutative differential graded R -algebra* (DG R -algebra, for short), that is,

- (a) $A = \bigoplus_{n \geq 0} A_n$ is a non-negatively graded commutative R -algebra¹, i.e., for all homogeneous elements $a, b \in A$ we have $ab = (-1)^{|a||b|}ba$, and $a^2 = 0$ if the degree of a (denoted $|a|$) is odd;
- (b) A is an R -complex with a differential d^A (that is, a graded R -linear map $A \rightarrow A$ of degree -1 with $(d^A)^2 = 0$); such that
- (c) d^A satisfies the *Leibniz rule*: for all homogeneous elements $a, b \in A$ the equality $d^A(ab) = d^A(a)b + (-1)^{|a|}ad^A(b)$ holds.

A *homomorphism* $f: A \rightarrow B$ of DG R -algebras is a graded R -algebra homomorphism of degree 0 which is also a chain map, that is, $d^B f = f d^A$.

2.2. An R -algebra U is a *divided power algebra* if a sequence of elements $u^{(i)} \in U$ with $i \in \mathbb{N} \cup \{0\}$ is correspondent to every element $u \in U$ with $|u|$ positive and even such that the following conditions are satisfied:

- (1) $u^{(0)} = 1$, $u^{(1)} = u$, and $|u^{(i)}| = i|u|$ for all i ;
- (2) $u^{(i)}u^{(j)} = \binom{i+j}{i}u^{(i+j)}$ for all i, j ;
- (3) $(u+v)^{(i)} = \sum_j u^{(j)}v^{(i-j)}$ for all i ;
- (4) for all $i \geq 2$ we have

$$(vw)^{(i)} = \begin{cases} 0 & |v| \text{ and } |w| \text{ are odd} \\ v^i w^{(i)} & |v| \text{ is even and } |w| \text{ is even and positive} \end{cases}$$

- (5) For all $i \geq 1$ and $j \geq 0$ we have

$$\left(u^{(i)}\right)^{(j)} = \frac{(ij)!}{j!(i!)^j}u^{(ij)}.$$

A *divided power DG R -algebra* is a DG R -algebra whose underlying graded R -algebra is a divided power algebra.

2.3. If R contains the field of rational numbers and U is a graded R -algebra, then U has a structure of a divided power R -algebra by defining $u^{(m)} = (1/m!)u^m$ for all $u \in U$ and integers $m \geq 0$; see [13, Lemma 1.7.2]. Also, R considered as a graded R -algebra concentrated in degree 0 is a divided power R -algebra.

2.4. Let $t \in A$ be a cycle, and let $A\langle X \rangle$ with the differential d denote the *simple free extension of A* obtained by adjunction of a variable X of degree $|t| + 1$ such that $dX = t$. The DG R -algebra $A\langle X \rangle$ can be described as $A\langle X \rangle = \bigoplus_{m \geq 0} X^{(m)}A$

¹Some authors use the cohomological notation for DG algebras. In such a case, A is described as $A = \bigoplus_{n \leq 0} A^n$, where $A^n = A_{-n}$ and A is called non-positively graded.

with the conventions $X^{(0)} = 1$ and $X^{(1)} = X$, where $\{X^{(m)} \mid m \geq 0\}$ is a free basis of $A\langle X \rangle$ such that:

(a) If $|X|$ is odd, then $X^{(m)} = 0$ for all $m \geq 2$, and for all $a + Xb \in A\langle X \rangle$ we have

$$d(a + Xb) = d^A a + tb - Xd^A b.$$

(b) If $|X|$ is even, then $A\langle X \rangle$ is a divided power DG R -algebra with the algebra structure given by $X^{(m)}X^{(\ell)} = \binom{m+\ell}{m}X^{(m+\ell)}$ and the differential structure defined by $dX^{(m)} = X^{(m-1)}t$ for all $m \geq 1$.

Also, let $A[X]$ denote the *simple polynomial extension* of A with X described as above, that is, $A[X] = \bigoplus_{m \geq 0} X^m A$ with $d^{A[X]}(X^m) = mX^{m-1}t$ for positive integers m . Note that here X^m is just the ordinary power on X .

If R contains the field of rational numbers, then $A\langle X \rangle = A[X]$.

2.5. Let n be a positive integer, and let $A\langle X_1, \dots, X_n \rangle$ (which is also denoted by $A\langle X_i \mid 1 \leq i \leq n \rangle$) be a *finite free extension of the DG R -algebra A* obtained by adjunction of n variables. In fact, setting $A^{(0)} = A$ and $A^{(i)} = A^{(i-1)}\langle X_i \rangle$ for all $1 \leq i \leq n$ such that $d^{A^{(i)}}X_i$ is a cycle in $A^{(i-1)}$, we have $A\langle X_1, \dots, X_n \rangle = A^{(n)}$. We also assume that $0 < |X_1| \leq \dots \leq |X_n|$. Note that there is a sequence of DG R -algebras $A = A^{(0)} \subset A^{(1)} \subset \dots \subset A^{(n)} = A\langle X_1, \dots, X_n \rangle$.

In a similar way, one can define the *finite polynomial extension of the DG R -algebra A* , which is denoted by $A[X_1, \dots, X_n]$.

2.6. Our discussion in 2.5 can be extended to the case of adjunction of infinitely countably many variables to the DG R -algebra A . Let $\{X_i \mid i \in \mathbb{N}\}$ be a set of variables. Attaching a degree to each variable such that $0 < |X_1| \leq |X_2| \leq \dots$, similar to 2.5, we construct a sequence $A = A^{(0)} \subset A^{(1)} \subset A^{(2)} \subset \dots$ of DG R -algebras. We define an *infinite free extension of the DG R -algebra A* obtained by adjunction of the variables X_1, X_2, \dots to be $A\langle X_i \mid i \in \mathbb{N} \rangle = \bigcup_{n \in \mathbb{N}} A^{(n)}$. It is sometimes convenient for us to use the notation $A\langle X_1, \dots, X_n \rangle$ with $n = \infty$ instead of $A\langle X_i \mid i \in \mathbb{N} \rangle$.

For the infinite extension $A\langle X_i \mid i \in \mathbb{N} \rangle$ of the DG R -algebra A , we always assume the *degree-wise finiteness condition*, that is, for all $n \in \mathbb{N}$, we assume that the set $\{i \mid |X_i| = n\}$ is finite. As an example of this situation, let $R \rightarrow S$ be a surjective ring homomorphism of commutative noetherian rings. Then the Tate resolution of S over R is an extension of the DG R -algebra R (with infinitely countably many variables, in general) which satisfies the degree-wise finiteness condition; see [32].

In a similar way, one can define the *infinite polynomial extension of the DG R -algebra A* , which is denoted by $A[X_i \mid i \in \mathbb{N}]$ or $A[X_1, \dots, X_n]$ with $n = \infty$.

2.7. For $n \leq \infty$, let $\Gamma = \bigcup_{i=1}^n \{X_i^{(m)} \mid m \geq 0\}$ with the conventions from 2.4 that if $|X_i|$ is odd, then $X_i^{(0)} = 1$, $X_i^{(1)} = X_i$, and $X_i^{(m)} = 0$ for all $m \geq 2$.

If $n < \infty$, then the set $\{X_1^{(m_1)}X_2^{(m_2)} \dots X_n^{(m_n)} \mid X_i^{(m_i)} \in \Gamma (1 \leq i \leq n)\}$ is a basis for the underlying graded free A -module $A\langle X_1, \dots, X_n \rangle$.

If $n = \infty$, then the set $\{X_{i_1}^{(m_{i_1})}X_{i_2}^{(m_{i_2})} \dots X_{i_t}^{(m_{i_t})} \mid X_{i_j}^{(m_{i_j})} \in \Gamma (i_j \in \mathbb{N}, t < \infty)\}$ is a basis for the underlying graded free A -module $A\langle X_i \mid i \in \mathbb{N} \rangle$.

The cases $A[X_1, \dots, X_n]$ and $A[X_i \mid i \in \mathbb{N}]$ can be treated similarly by using ordinary powers X_i^m instead of divided powers $X_i^{(m)}$.

2.8. A right DG A -module (M, ∂^M) (or simply M) is a graded right A -module $M = \bigoplus_{i \in \mathbb{Z}} M_i$ that is also an R -complex with the differential ∂^M satisfying the Leibniz rule, that is, the equality $\partial^M(ma) = \partial^M(m)a + (-1)^{|m|}m d^A(a)$ holds for all homogeneous elements $a \in A$ and $m \in M$.

All DG modules considered in this paper are right DG modules, unless otherwise stated. Since A is graded commutative, a DG A -module M is also a left DG A -module with the left A -action defined by $am = (-1)^{|m||a|}ma$ for $a \in A$ and $m \in M$.

A DG submodule of a DG A -module M is a subcomplex that is a DG A -module under the operations induced by M , and a DG ideal of A is a DG submodule of A .

For a DG A -module M , let $\inf(M) = \inf\{i \in \mathbb{Z} \mid M_i \neq 0\}$. We say that M is *bounded below* if $\inf(M) > -\infty$, that is, if $M_i = 0$ for all $i \ll 0$. Note that $\inf(L) \geq \inf(M)$ if L is a DG A -submodule of M . For an integer i , the i -th *shift* of M , denoted $\Sigma^i M$ or $M(-i)$, is defined by $(\Sigma^i M)_j = M_{j-i}$ with $\partial_j^{\Sigma^i M} = (-1)^i \partial_{j-i}^M$.

2.9. Let A° denote the *opposite DG R -algebra* which is equal to A as a set, but to distinguish elements in A° and A we write $a^\circ \in A^\circ$ if $a \in A$. The product of elements in A° and the differential d^{A° are given by the formulas $a^\circ b^\circ = (-1)^{|a||b|}(ba)^\circ = (ab)^\circ$ and $d^{A^\circ}(a^\circ) = d^A(a)^\circ$, for all homogeneous elements $a, b \in A$. Since A is a graded commutative DG R -algebra, the identity map $A \rightarrow A^\circ$ that corresponds $a \in A$ to $a^\circ \in A^\circ$ is a DG R -algebra isomorphism. From this point of view, there is no need to distinguish between A and A° . However, we will continue using the notation A° to make it clear how we use the graded commutativity of A .

Note that every right (resp. left) DG A -module M is a left (resp. right) DG A° -module with $a^\circ m = (-1)^{|a^\circ||m|}ma$ (resp. $ma^\circ = (-1)^{|a^\circ||m|}am$) for all homogeneous elements $a \in A$ and $m \in M$.

2.10. Let $A \rightarrow B$ be a homomorphism of DG R -algebras such that B is projective as an underlying graded A -module. Let B^e denote the *enveloping DG R -algebra* $B^\circ \otimes_A B$ of B over A . The algebra structure on B^e is given by

$$(b_1^\circ \otimes b_2)(b_1^\circ \otimes b_2) = (-1)^{|b_1^\circ||b_2|} b_1^\circ b_1^\circ \otimes b_2 b_2 = (-1)^{|b_1^\circ||b_2| + |b_1^\circ||b_1|} (b_1^\circ b_1)^\circ \otimes b_2 b_2$$

for all homogeneous elements $b_1, b_2, b_1^\circ, b_2^\circ \in B$, while the graded structure is given by $(B^e)_i = \sum_j (B^\circ)_j \otimes_A B_{i-j}$ and the differential d^{B^e} is defined by $d^{B^e}(b_1^\circ \otimes b_2) = d^{B^\circ}(b_1^\circ) \otimes b_2 + (-1)^{|b_1^\circ|} b_1^\circ \otimes d^B(b_2)$.

Note that B and B° are regarded as subrings of B^e . Moreover, the map $B^\circ \rightarrow B^e$ defined by $b^\circ \mapsto b^\circ \otimes 1$ is an injective DG R -algebra homomorphism, via which we can consider B° as a DG R -subalgebra of B^e . Since B is graded commutative, $B \cong B^\circ$ and hence, B is a DG R -subalgebra of B^e as well.

Note also that DG B^e -modules are precisely DG (B, B) -bimodules. In fact, for a DG B^e -module N , the right action of an element of B^e on N yields the two-sided module structure $n(b_1^\circ \otimes b_2) = (-1)^{|b_1^\circ||n|} b_1 n b_2$ for all homogeneous elements $n \in N$ and $b_1, b_2 \in B$. Hence, the differential ∂^N satisfies the Leibniz rule on both sides: $\partial^N(b_1 n b_2) = d^B(b_1) n b_2 + (-1)^{|b_1|} b_1 \partial^N(n) b_2 + (-1)^{|b_1| + |n|} b_1 n d^B(b_2)$ for all homogeneous elements $n \in N$ and $b_1, b_2 \in B$.

2.11. Consider the notation from 2.7 and 2.10. Let

$$\begin{aligned} \text{Mon}(\Gamma) &= \begin{cases} \{(1^\circ \otimes X_1^{(m_1)}) \cdots (1^\circ \otimes X_n^{(m_n)}) \mid X_i^{(m_i)} \in \Gamma \ (1 \leq i \leq n)\} & \text{if } n < \infty \\ \{(1^\circ \otimes X_{i_1}^{(m_{i_1})}) \cdots (1^\circ \otimes X_{i_t}^{(m_{i_t})}) \mid X_{i_j}^{(m_{i_j})} \in \Gamma \ (i_j \in \mathbb{N}, t < \infty)\} & \text{if } n = \infty \end{cases} \\ &= \begin{cases} \{1^\circ \otimes (X_1^{(m_1)} \cdots X_n^{(m_n)}) \mid X_i^{(m_i)} \in \Gamma \ (1 \leq i \leq n)\} & \text{if } n < \infty \\ \{1^\circ \otimes (X_{i_1}^{(m_{i_1})} \cdots X_{i_t}^{(m_{i_t})}) \mid X_{i_j}^{(m_{i_j})} \in \Gamma \ (i_j \in \mathbb{N}, t < \infty)\} & \text{if } n = \infty \end{cases} \end{aligned}$$

Then the underlying graded $A\langle X_1, \dots, X_n \rangle^\circ$ -module $A\langle X_1, \dots, X_n \rangle^e$ with $n \leq \infty$ is free with the basis $\text{Mon}(\Gamma)$.²

Once again, the case of $A[X_1, \dots, X_n]$ with $n \leq \infty$ can be treated similarly by using X_i^m instead of $X_i^{(m)}$.

2.12. A *semifree basis* (or *semi-basis*) of a DG A -module M is a well-ordered subset $F \subseteq M$ that is a basis for the underlying graded A -module M and satisfies $\partial^M(f) \in \sum_{e < f} eA$ for every element $f \in F$. A DG A -module M is *semifree*³ if it has a semifree basis. Equivalently, the DG A -module M is semifree if there exists an increasing filtration

$$0 = F_{-1} \subseteq F_0 \subseteq F_1 \subseteq \cdots \subseteq M$$

of DG A -submodules of M such that $M = \bigcup_{i \geq 0} F_i$ and each DG A -module F_i/F_{i-1} is a direct sum of copies of $A(n)$ with $n \in \mathbb{Z}$; see [8], [9, A.2], or [12].

2.13. Let $\mathcal{C}(A)$ denote the abelian category of DG A -modules and DG A -module homomorphisms. Also, let $\mathcal{K}(A)$ be the *homotopy category* of DG A -modules. Recall that objects of $\mathcal{K}(A)$ are DG A -modules and morphisms are the set of homotopy equivalence classes of DG A -module homomorphisms $\text{Hom}_{\mathcal{K}(A)}(M, L) = \text{Hom}_{\mathcal{C}(A)}(M, L)/\sim$, where $f \sim g$ for $f, g \in \text{Hom}_{\mathcal{C}(A)}(M, L)$ if and only if there is a graded A -module homomorphism $h: M \rightarrow L(-1)$ of underlying graded A -modules such that $f - g = \partial^L h + h\partial^M$. It is known that $\mathcal{K}(A)$ is triangulated category. In fact, there is a triangle $M \rightarrow L \rightarrow Z \rightarrow \Sigma M$ in $\mathcal{K}(A)$ if and only if there is a short exact sequence $0 \rightarrow M \rightarrow L \oplus L' \rightarrow Z \rightarrow 0$ in $\mathcal{C}(A)$ in which L' is splitting exact, i.e., $\text{id}_{L'} \sim 0$. The *derived category* $\mathcal{D}(A)$ is obtained from $\mathcal{C}(A)$ by formally inverting the quasi-isomorphisms (denoted \simeq); see, for instance, [18] for details.

For each integer i and DG A -modules M, L with M being semifree, $\text{Ext}_A^i(M, L)$ is defined to be $H_{-i}(\text{Hom}_A(M, L))$. Note that $\text{Ext}_A^i(M, L) = \text{Hom}_{\mathcal{K}(A)}(M, L(-i))$.

3. DIAGONAL IDEALS AND DG SMOOTHNESS

In this section, we introduce the notion of diagonal ideals which play an essential role in the proofs of Theorem 1.2 and Main Theorem.

3.1. Let $\varphi: A \rightarrow B$ be a homomorphism of DG R -algebras such that B is projective as an underlying graded A -module. Let $\pi_B: B^e \rightarrow B$ denote the map defined by

²“Mon” is chosen for “monomial.”

³Keller [18] calls these “DG modules that have property (P).”

$\pi_B(b_1^o \otimes b_2) = b_1 b_2$. For all homogeneous elements $b_1, b_2, b'_1, b'_2 \in B$ we have

$$\begin{aligned} \pi_B((b_1^o \otimes b_2)(b'_1{}^o \otimes b'_2)) &= (-1)^{|b'_1||b_2|+|b'_1||b_1|} \pi_B((b'_1 b_1)^o \otimes b_2 b'_2) \\ &= (-1)^{|b'_1||b_2|+|b'_1||b_1|} (b'_1 b_1)(b_2 b'_2) \\ &= (b_1 b_2)(b'_1 b'_2) \\ &= \pi_B(b_1^o \otimes b_2) \pi_B(b'_1{}^o \otimes b'_2). \end{aligned}$$

Hence, π_B is an algebra homomorphism. Also, it is straightforward to check that π_B is a chain map. Therefore, π_B is a homomorphism of DG R -algebras.

Definition 3.2. In the setting of 3.1, kernel of π_B is denoted by $J = J_{B/A}$ and is called the *diagonal ideal* of φ .⁴

3.3. In 3.1, since π_B is a homomorphism of DG R -algebras, J is a DG ideal of B^e . The isomorphism $B^e/J \cong B$ of DG R -algebras is also an isomorphism of DG B^e -modules. Hence, there is an exact sequence of DG B^e -modules:

$$0 \rightarrow J \rightarrow B^e \xrightarrow{\pi_B} B \rightarrow 0. \quad (3.3.1)$$

Next, we define our notion of smoothness for DG algebras.

Definition 3.4. Let $\varphi: A \rightarrow B$ be a homomorphism of DG R -algebras. We say that B is *DG quasi-smooth over A* (or simply *φ is DG quasi-smooth*) if the following conditions are satisfied:

- (i) B is free as an underlying graded A -module.
- (ii) The diagonal ideal J has a filtration consisting of DG B^e -submodules⁵

$$J = J^{[1]} \supset J^{[2]} \supset J^{[3]} \supset \dots \supset J^{[\ell]} \supset J^{[\ell+1]} \supset \dots$$

such that $JJ^{[\ell]} + J^{[\ell]}J \subseteq J^{[\ell+1]}$ for all $\ell \geq 1$, and each element of $J^{[\ell]}$ has degree $\geq \ell$, that is, $\inf(J^{[\ell]}) \geq \ell$. This implies that $\bigcap_{\ell} J^{[\ell]} = (0)$.

- (iii) For every $\ell \geq 1$, the DG B -module $J^{[\ell]}/J^{[\ell+1]}$ is semifree.

We say that B is *DG smooth over A* (or simply *φ is DG smooth*) if it is DG quasi-smooth over A and for all positive integers ℓ , the semifree DG B -module $J^{[\ell]}/J^{[\ell+1]}$ has a finite semifree basis.⁶

3.5. There exist other definitions of smoothness for DG algebras. For instance, a definition given by Kontsevich is found in [20] (alternatively, in [34, Section 18]). Also, another version of smoothness for DG algebras is introduced by Toën and Vezzosi in [33] which Shaul [31] proves is equivalent to Kontsevich's definition. However, our above version of smoothness is new and quite different from any existing definition of smoothness for DG algebras.

3.6. Let $\varphi: A \rightarrow B$ be a homomorphism of DG R -algebras. If B is DG smooth over A , then for any integer $\ell \geq 1$, there is a finite filtration

$$J = L_0 \supset L_1 \supset L_2 \supset \dots \supset L_s \supset L_{s+1} = J^{[\ell]}$$

⁴The definition of diagonal ideals originates in scheme theory. In fact, if $A \rightarrow B$ is a homomorphism of commutative rings, then the kernel of the natural mapping $B \otimes_A B \rightarrow B$ is the defining ideal of the diagonal set in the Cartesian product $\text{Spec } B \times_{\text{Spec } A} \text{Spec } B$.

⁵ $J^{[\ell]}$ is just a notation for the ℓ -th DG B^e -submodule of J in the sequence. It is not an ℓ -th power of any kind.

⁶In case that $A \rightarrow B$ is a homomorphism of commutative rings, B is projective over A , and J/J^2 is projective over B , then B is smooth over A in the sense of scheme theory. In this case, $J/J^2 \cong \Omega_{B/A}$ is the module of Kähler differentials.

of J by its DG B^e -submodules, where for each $0 \leq i \leq s$ we have $L_i/L_{i+1} \cong B(-a_i)$ as DG B^e -modules, for some positive integer a_i .

3.7. We will show in Section 6 that free extensions of divided power DG R -algebras and polynomial extensions of DG R -algebras are DG quasi-smooth. If these extensions are finite, then we have the DG smooth property; see Corollary 6.12.

There are several examples of DG smooth extensions besides free or polynomial extensions. For instance, as one of the most trivial examples, let $B = A\langle X \rangle / (X^2)$, where $|X|$ is even and $d^B X = 0$. If A contains a field of characteristic 2, then B is DG smooth over A by setting $J^{[2]} = (0)$.

3.8. Let $\varphi: A \rightarrow B$ be a DG quasi-smooth homomorphism, and use the notation of Definition 3.4. Let N be a semifree DG B -module. For every positive integer ℓ , consider the DG B^e -module $J/J^{[\ell]}$ as a DG (B, B) -bimodule. The tensor product $N \otimes_B J/J^{[\ell]}$ uses the left DG B -module structure of $J/J^{[\ell]}$ and in this situation, $N \otimes_B J/J^{[\ell]}$ is a right DG B -module by the right B -action on $J/J^{[\ell]}$.

The following lemma is useful in the next section.

Lemma 3.9. *Let $\varphi: A \rightarrow B$ be a DG quasi-smooth homomorphism, and use the notation of Definition 3.4. Suppose that N is a semifree DG B -module such that $\text{Ext}_B^i(N, N \otimes_B J/J^{[\ell]}) = 0$ for all $i \geq 0$ and some $\ell \geq 1$. Then the natural inclusion $J^{[\ell]} \hookrightarrow J$ induces an isomorphism $\text{Ext}_B^i(N, N \otimes_B J^{[\ell]}) \cong \text{Ext}_B^i(N, N \otimes_B J)$ for all $i \geq 1$.*

Proof. Applying $N \otimes_B -$ to the short exact sequence $0 \rightarrow J^{[\ell]} \rightarrow J \rightarrow J/J^{[\ell]} \rightarrow 0$ of DG B^e -modules, we get an exact sequence

$$0 \rightarrow N \otimes_B J^{[\ell]} \rightarrow N \otimes_B J \rightarrow N \otimes_B J/J^{[\ell]} \rightarrow 0 \quad (3.9.1)$$

of DG B -modules, where the injectivity on the left comes from the fact that N is free as an underlying graded B -module. The assertion now follows from the long exact sequence of Ext obtained from applying $\text{Hom}_B(N, -)$ to (3.9.1). \square

The following result is used in the proof of Theorem 1.2.

Theorem 3.10. *Let $\varphi: A \rightarrow B$ be a DG smooth homomorphism, and use the notation of Definition 3.4. Let N be a semifree DG B -module with $\text{Ext}_B^i(N, N) = 0$ for all $i \geq 1$. Then for all $i \geq 0$ and all $\ell \geq 1$ we have $\text{Ext}_B^i(N, N \otimes_B J^{[\ell]}/J^{[\ell+1]}) = 0 = \text{Ext}_B^i(N, N \otimes_B J/J^{[\ell]})$.*

Proof. We treat both of the equalities at the same time. Let L denote $J^{[\ell]}/J^{[\ell+1]}$ or $J/J^{[\ell]}$ with $\ell \geq 1$. By definition of DG smoothness, there is a finite filtration

$$L = L_0 \supset L_1 \supset L_2 \supset \cdots \supset L_s \supset L_{s+1} = (0)$$

of L by its DG B^e -submodules, where for each $0 \leq i \leq s$ we have $L_i/L_{i+1} \cong B(-a_i)$ as DG B^e -modules, for some positive integer a_i .

We now prove by induction on s that $\text{Ext}_B^i(N, N \otimes_B L) = 0$ for all $i \geq 0$. For the base case where $s = 0$, we have $L \cong B(-a_0)$. Hence, $N \otimes_B L \cong N(-a_0)$. Therefore, $\text{Ext}_B^i(N, N \otimes_B L) \cong \text{Ext}_B^i(N, N(-a_0)) = \text{Ext}_B^{i+a_0}(N, N) = 0$ for all $i \geq 0$.

Assume now that $s \geq 1$. Since N is a semifree DG B -module, tensoring the short exact sequence $0 \rightarrow L_1 \rightarrow L \rightarrow B(-a_0) \rightarrow 0$ of DG B^e -modules (hence, DG B -modules) by N , we get a short exact sequence

$$0 \rightarrow N \otimes_B L_1 \rightarrow N \otimes_B L \rightarrow N \otimes_B B(-a_0) \rightarrow 0 \quad (3.10.1)$$

of DG B -modules (hence, DG B^c -modules via π_B). By inductive hypothesis we have $\text{Ext}_B^i(N, N \otimes_B L_1) = 0$ for all $i \geq 0$. Also, $\text{Ext}_B^i(N, N \otimes_B B(-a_0)) = 0$ for all $i \geq 0$ by the base case $s = 0$. It follows from the long exact sequence of cohomology modules obtained from (3.10.1) that $\text{Ext}_B^i(N, N \otimes_B L) = 0$ for all $i \geq 0$. \square

The next result is also crucial in the proof of Theorem 1.2.

Theorem 3.11. *Let $\varphi: A \rightarrow B$ be a DG quasi-smooth homomorphism, and use the notation of Definition 3.4. Let N be a bounded below semifree DG B -module with $\text{Ext}_B^i(N, N \otimes_B J/J^{[\ell]}) = 0$ for all $i \geq 0$ and all $\ell \geq 1$. Then $\text{Ext}_B^i(N, N \otimes_B J) = 0$ for all $i \geq 1$.*

The proof of this result needs the machinery of homotopy limits, which we discuss in the next section. We give the proof of this theorem in 4.7 below.

4. HOMOTOPY LIMITS AND PROOF OF THEOREM 3.11

The entire section is devoted to the proof of Theorem 3.11. The notion of homotopy limits, which we define in 4.4, plays an essential role in the proof of the following result which is a key to the proof of Theorem 3.11.

Theorem 4.1. *Let M and N be DG A -modules with M bounded below and N semifree. Assume that there is a descending sequence*

$$M = M^0 \supseteq M^1 \supseteq M^2 \supseteq \dots \supseteq M^\ell \supseteq M^{\ell+1} \supseteq \dots$$

of DG A -submodules of M that satisfies the following conditions:

- (1) $\lim_{\ell \rightarrow \infty} \inf(M^\ell) = \infty$; and
- (2) *there is an integer k such that the natural maps $M^\ell \hookrightarrow M$ induce isomorphisms $\text{Ext}_A^k(N, M^\ell) \cong \text{Ext}_A^k(N, M)$ for all $\ell \geq 1$.*

Then $\text{Ext}_A^k(N, M) = 0$.

By 2.13, Theorem 4.1 can be restated as follows. We prove this result in 4.6.

Theorem 4.2. *Let M and N be DG A -modules with M bounded below and N semifree. Assume that there is a descending sequence*

$$M = M^0 \supseteq M^1 \supseteq M^2 \supseteq \dots \supseteq M^\ell \supseteq M^{\ell+1} \supseteq \dots$$

of DG A -submodules of M that satisfies the following conditions:

- (1) $\lim_{\ell \rightarrow \infty} \inf(M^\ell) = \infty$; and
- (2) *for all positive integers ℓ , the natural maps $M^\ell \hookrightarrow M$ induce isomorphisms $\text{Hom}_{\mathcal{K}(A)}(N, M^\ell) \cong \text{Hom}_{\mathcal{K}(A)}(N, M)$.*

Then $\text{Hom}_{\mathcal{K}(A)}(N, M) = 0$.

4.3. For a family $\{M^\ell \mid \ell \in \mathbb{N}\}$ of countably many DG A -modules, the *product* (or the *direct product*) $P = \prod_{\ell \in \mathbb{N}} M^\ell$ in $\mathcal{C}(A)$ is constructed as follows: the DG A -module P has a \mathbb{Z} -graded structure $P_i = \prod_{\ell \in \mathbb{N}} (M^\ell)_i$ for all $i \in \mathbb{Z}$ with the differential that is given by the formula

$$\partial_i^P((m^\ell)_{\ell \in \mathbb{N}}) = \left(\partial_i^{M^\ell}(m^\ell) \right)_{\ell \in \mathbb{N}}$$

for all $(m^\ell)_{\ell \in \mathbb{N}} \in P_i$. By definition, we have

$$\text{Hom}_{\mathcal{C}(A)}(-, P) \cong \prod_{\ell \in \mathbb{N}} \text{Hom}_{\mathcal{C}(A)}(-, M^\ell)$$

as functors on $\mathcal{C}(A)$. It can be seen that P is also a product in $\mathcal{K}(A)$. Hence,

$$\mathrm{Hom}_{\mathcal{K}(A)}(-, P) \cong \prod_{\ell \in \mathbb{N}} \mathrm{Hom}_{\mathcal{K}(A)}(-, M^\ell) \quad (4.3.1)$$

as functors on $\mathcal{K}(A)$.

Next, we define the notion of homotopy limits; references on this include [7, 18].

4.4. Assume that $\{M^\ell \mid \ell \in \mathbb{N}\}$ is a family of DG A -modules such that

$$M^1 \supseteq M^2 \supseteq \cdots \supseteq M^\ell \supseteq M^{\ell+1} \supseteq \cdots.$$

Then, the *homotopy limit* $L = \mathrm{holim} M^\ell$ is defined by the triangle in $\mathcal{K}(A)$

$$L \rightarrow P \xrightarrow{\varphi} P \rightarrow \Sigma L \quad (4.4.1)$$

where $P = \prod_{\ell \in \mathbb{N}} M^\ell$ is the product introduced in 4.3 and φ is defined by

$$\varphi((m^\ell)_{\ell \in \mathbb{N}}) = (m^\ell - m^{\ell+1})_{\ell \in \mathbb{N}}.$$

Note that (4.4.1) is a triangle in the derived category $\mathcal{D}(A)$ as well.

Let N be a semifree DG A -module. Since N is $\mathcal{K}(A)$ -projective, we note that $\mathrm{Hom}_{\mathcal{D}(A)}(N, -) = \mathrm{Hom}_{\mathcal{K}(A)}(N, -)$ on the object set of $\mathcal{C}(A)$. Applying the functor $\mathrm{Hom}_{\mathcal{K}(A)}(N, -)$ to the triangle (4.4.1), by 4.3 we have a triangle

$$\mathrm{Hom}_{\mathcal{K}(A)}(N, L) \rightarrow \prod_{\ell \in \mathbb{N}} \mathrm{Hom}_{\mathcal{K}(A)}(N, M^\ell) \rightarrow \prod_{\ell \in \mathbb{N}} \mathrm{Hom}_{\mathcal{K}(A)}(N, M^\ell) \rightarrow \Sigma \mathrm{Hom}_{\mathcal{K}(A)}(N, L)$$

in $\mathcal{D}(R)$. Therefore, $\mathrm{Hom}_{\mathcal{K}(A)}(N, \mathrm{holim} M^\ell) \cong \mathrm{holim} \mathrm{Hom}_{\mathcal{K}(A)}(N, M^\ell)$ in $\mathcal{D}(R)$.

Lemma 4.5. *Under the assumptions of Theorem 4.2 we have $\mathrm{H}(\mathrm{holim} M^\ell) = 0$, that is, $\mathrm{holim} M^\ell$ is zero in the derived category $\mathcal{D}(A)$.*

Proof. Let $L = \mathrm{holim} M^\ell$. The triangle (4.4.1) gives the long exact sequence

$$\cdots \rightarrow \mathrm{H}_i(L) \rightarrow \prod_{\ell \in \mathbb{N}} \mathrm{H}_i(M^\ell) \xrightarrow{\mathrm{H}(\varphi)} \prod_{\ell \in \mathbb{N}} \mathrm{H}_i(M^\ell) \rightarrow \mathrm{H}_{i-1}(L) \rightarrow \cdots$$

of homology modules. Fix an integer i and note that $\mathrm{H}_i(M^\ell) = 0$ if $\inf(M^\ell) > i$. Hence, by Condition (1) we have $\mathrm{H}_i(M^\ell) = 0$ for almost all $\ell \in \mathbb{N}$. Thus, the product $\prod_{\ell \in \mathbb{N}} \mathrm{H}_i(M^\ell)$ is a product of finite number of R -modules. Hence, $\mathrm{H}(\varphi)$ is an isomorphism and therefore, $\mathrm{H}_i(L) = 0$ for all $i \in \mathbb{Z}$, as desired. \square

4.6. *Proof of Theorem 4.2.* Since N is semifree, by 4.4 and Condition (2)

$$\begin{aligned} \mathrm{Hom}_{\mathcal{K}(A)}(N, M) &\cong \mathrm{holim} \mathrm{Hom}_{\mathcal{K}(A)}(N, M^\ell) \\ &\cong \mathrm{Hom}_{\mathcal{K}(A)}(N, \mathrm{holim} M^\ell) \\ &\cong \mathrm{Hom}_{\mathcal{D}(A)}(N, \mathrm{holim} M^\ell). \end{aligned}$$

Now the assertion follows from Lemma 4.5. \square

4.7. *Proof of Theorem 3.11.* It follows from Lemma 3.9 that $\mathrm{Ext}_B^i(N, N \otimes_B J^{[\ell]}) \cong \mathrm{Ext}_B^i(N, N \otimes_B J)$ for all $i \geq 1$ and all $\ell \geq 1$. Note that $\{\inf(N \otimes_B J^{[\ell]}) \mid \ell \in \mathbb{N}\}$ is an increasing sequence of integers that diverges to ∞ . Now, the assertion follows from Theorem 4.1. \square

5. NAÏVE LIFTABILITY AND PROOF OF THEOREM 1.2

The notion of naïve liftability was introduced by the authors in [21] along simple free extensions of DG algebras. It is shown in [21, Theorem 6.8] that along such extension $A \rightarrow A\langle X \rangle$ of DG algebras, weak liftability in the sense of [21, Definition 5.1] (when $|X|$ is odd) and liftability (when $|X|$ is even) of DG modules are equivalent to naïve liftability. In this section, we study the naïve lifting property of DG modules in a more general setting using the diagonal ideal. We give the proof of Theorem 1.2 in 5.7.

5.1. Let $A \rightarrow B$ be a homomorphism of DG R -algebras such that B is free as an underlying graded A -module. Let (N, ∂^N) be a semifree DG B -module, and let $N|_A$ denote N regarded as a DG A -module via $A \rightarrow B$. Since B is free as an underlying graded A -module, $N|_A$ is a semifree DG A -module. Note that $(N|_A \otimes_A B, \partial)$ is a DG B -module with $\partial(n \otimes b) = \partial^N(n) \otimes b + (-1)^{|n|} n \otimes d^B(b)$ for all homogeneous elements $n \in N$ and $b \in B$. Since $N|_A$ is a semifree DG A -module, $N|_A \otimes_A B$ is a semifree DG B -module, and we have a (right) DG B -module epimorphism $\pi_N: N|_A \otimes_A B \rightarrow N$ defined by $\pi_N(n \otimes b) = nb$.

Proposition 5.2. *Let $A \rightarrow B$ be a homomorphism of DG R -algebras such that B is free as an underlying graded A -module. Every semifree DG B -module N fits into the following short exact sequence of DG B -modules:*

$$0 \rightarrow N \otimes_B J \rightarrow N|_A \otimes_A B \xrightarrow{\pi_N} N \rightarrow 0. \quad (5.2.1)$$

Proof. Since the left DG B -module B^o is the right DG B -module B , we have an isomorphism $N \otimes_B B^o \cong N$ of right DG A -modules such that $x \otimes b^o \mapsto xb$ for all $x \in N$ and $b \in B$. Hence, there are isomorphisms $N \otimes_B B^e = N \otimes_B (B^o \otimes_A B) \cong (N \otimes_B B^o) \otimes_A B \cong N|_A \otimes_A B$ such that $x \otimes (b_1^o \otimes b_2) \mapsto xb_1 \otimes b_2$ for all $x \in N$ and $b_1, b_2 \in B$. Therefore, we get the commutative diagram

$$\begin{array}{ccc} N \otimes_B B^e & \xrightarrow{\text{id}_N \otimes \pi_B} & N \otimes_B B \\ \cong \downarrow & & \downarrow \cong \\ N|_A \otimes_A B & \xrightarrow{\pi_N} & N \end{array} \quad (5.2.2)$$

of DG B -module homomorphisms. Thus, by applying $N \otimes_B -$ to the short exact sequence (3.3.1), we obtain the short exact sequence (5.2.1) in which injectivity on the left follows from the fact that N is free as an underlying graded B -module. \square

5.3. To clarify, note that the DG algebra homomorphism $\pi_B: B^e \rightarrow B$ defined in 3.1 coincides with the DG algebra homomorphism $\pi_B: B|_A \otimes_A B \rightarrow B$ defined in 5.1. In fact, as we mentioned in the proof of Proposition 5.2 (the left column in (5.2.2) with $N = B$), we have the isomorphism $B^e \cong B|_A \otimes_A B$.

We remind the reader of the definition of naïve liftability from the introduction.

Definition 5.4. Let $A \rightarrow B$ be a homomorphism of DG R -algebras such that B is free as an underlying graded A -module. A semifree DG B -module N is *naïvely liftable* to A if the map π_N is a split DG B -module epimorphism, i.e., there exists a DG B -module homomorphism $\rho: N \rightarrow N|_A \otimes_A B$ that satisfies the equality $\pi_N \rho = \text{id}_N$. Equivalently, N is naïvely liftable to A if π_N has a right inverse in the abelian category of right DG B -modules.

If a semifree DG B -module N is naively liftable to A , then the short exact sequence (5.2.1) splits. This implies the following result.

Corollary 5.5. *Let $A \rightarrow B$ be a homomorphism of DG R -algebras such that B is free as an underlying graded A -module. If a semifree DG B -module N is naively liftable to A , then N is a direct summand of the DG B -module $N|_A \otimes_A B$ which is liftable to A .*

We use the following result in the proof of Theorem 1.2.

Theorem 5.6. *Let $A \rightarrow B$ be a homomorphism of DG R -algebras such that B is free as an underlying graded A -module. If N is a semifree DG B -module such that $\text{Ext}_B^1(N, N \otimes_B J) = 0$, then N is naively liftable to A .*

Proof. Since N is a semifree DG B -module, it follows from our Ext-vanishing assumption and [23, Theorem A] that the short exact sequence (5.2.1) splits. Hence, N is naively liftable to A , as desired. \square

5.7. Proof of Theorem 1.2. By Theorem 3.10 we have $\text{Ext}_B^i(N, N \otimes_B J/J^{[\ell]}) = 0$ for all $i \geq 0$ and all $\ell \geq 1$. It follows from Theorem 3.11 that $\text{Ext}_B^i(N, N \otimes_B J) = 0$ for all $i \geq 1$. Hence, by Theorem 5.6, N is naively liftable to A . The fact that N is a direct summand of a DG B -module that is liftable to A was already proved in Corollary 5.5. \square

6. DIAGONAL IDEALS IN FREE AND POLYNOMIAL EXTENSIONS AND PROOF OF MAIN THEOREM

This section is devoted to the properties of diagonal ideals in free and polynomial extensions of certain DG R -algebra A with the aim to prove that such extensions are DG (quasi-)smooth over A ; see Corollary 6.12. Then we will give the proof of our Main Theorem in 6.13. First, we focus on free extensions of DG algebras.

6.1. Let $B = A\langle X_1, \dots, X_n \rangle$ with $n \leq \infty$, where A is a divided power DG R -algebra. It follows from [13, Proposition 1.7.6] that B is a divided power DG R -algebra. Also, B° is a divided power DG R -algebra with the divided power structure $(b^\circ)^{(i)} = (b^{(i)})^\circ$, for all $b \in B$ and $i \in \mathbb{N}$.

The next lemma indicates that, in the setting of 6.1, the map π_B is a homomorphism of divided power algebras in the sense of [13, Definition 1.7.3].

Lemma 6.2. *Let $B = A\langle X_1, \dots, X_n \rangle$ with $n \leq \infty$, where A is a divided power DG R -algebra. The DG algebra homomorphism π_B preserves the divided powers.*

Proof. Note that B^e is a divided power DG R -algebra by setting

$$(b_1^\circ \otimes b_2)^(i) = \begin{cases} (b_1^i)^\circ \otimes b_2^{(i)} & \text{if } |b_1|, |b_2| \text{ are even and } |b_2| > 0 \\ 0 & \text{if } |b_1|, |b_2| \text{ are odd} \end{cases}$$

for all $b_1^o \otimes b_2 \in B^e$ of positive even degree and all integers $i \geq 2$. By the properties of divided powers in 2.1, for such $b_1^o \otimes b_2 \in B^e$ we have

$$\begin{aligned} \pi_B \left((b_1^o \otimes b_2)^{(i)} \right) &= \begin{cases} \pi_B \left((b_1^i)^o \otimes b_2^{(i)} \right) & \text{if } |b_1|, |b_2| \text{ are even and } |b_2| > 0 \\ 0 & \text{if } |b_1|, |b_2| \text{ are odd} \end{cases} \\ &= \begin{cases} b_1^i b_2^{(i)} & \text{if } |b_1|, |b_2| \text{ are even and } |b_2| > 0 \\ 0 & \text{if } |b_1|, |b_2| \text{ are odd} \end{cases} \\ &= (b_1 b_2)^{(i)} \\ &= (\pi_B(b_1^o \otimes b_2))^{(i)}. \end{aligned}$$

Now, the assertion follows from the fact that every element of B^e is a finite sum of the elements of the form $b_1^o \otimes b_2$. \square

6.3. In Lemma 6.2, we assume that A is a divided power DG R -algebra to show that J is closed under taking divided powers. Note that elements of J are not all of the form of a monomial, so to define the ‘‘powers’’ of non-monomial elements we need to consider the divided powers. For example, for a positive integer ℓ we cannot define $(X_i + a)^{(\ell)}$, where $a \in A$ without assuming that A is a divided power DG R -algebra.

6.4. Let $B = A\langle X_1, \dots, X_n \rangle$ with $n \leq \infty$, where A is a divided power DG R -algebra. For $1 \leq i \leq n$, the *diagonal of the variable* X_i is an element of B^e which is defined by the formula

$$\xi_i = X_i^o \otimes 1 - 1^o \otimes X_i. \quad (6.4.1)$$

Since $\pi_B(\xi_i) = 0$, we have that $\xi_i \in J$ for all i . Note that if $|X_i|$ is odd, then $\xi_i^2 = 0$. From the basic properties of divided powers we have

$$\xi_i^{(m)} = \sum_{j=0}^m (-1)^{m-j} \left(X_i^{(j)} \right)^o \otimes X_i^{(m-j)} \quad (6.4.2)$$

for all $m \in \mathbb{N}$, considering the conventions that $\xi_i^{(0)} = 1^o \otimes 1$ and $\xi_i^{(m)} = 0$ for all $m \geq 2$ if $|\xi_i| = |X_i|$ is odd. Note that $\xi_i^{(1)} = \xi_i$ by definition. Since by Lemma 6.2, the map π_B preserves the divide powers, we see that $\xi_i^{(m)} \in J$ for all i and $m \in \mathbb{N}$. Let $\Omega = \{\xi_i^{(m)} \mid 1 \leq i \leq n, m \in \mathbb{N}\}$.

Lemma 6.5. *Let $B = A\langle X_1, \dots, X_n \rangle$ with $n \leq \infty$, where A is a divided power DG R -algebra. The diagonal ideal J is generated by Ω , that is, $J = \Omega B^e$.*

The statement of this lemma is equivalent to the equality $J = B\Omega B$.

Proof. Since $\Omega \subseteq J$, we have $J' := \Omega B^e \subseteq J$. Now we show $J \subseteq J'$.

We claim that for all $1 \leq i \leq n$ with $n \leq \infty$ and all $m \in \mathbb{N}$ we have the equality

$$(X_i^{(m)})^o \otimes 1 \equiv 1^o \otimes X_i^{(m)} \pmod{J'}. \quad (6.5.1)$$

To prove this claim, we proceed by induction on $m \in \mathbb{N}$. For the base case, since $\xi_i \in \Omega \subseteq J'$, we have $X_i^o \otimes 1 \equiv 1^o \otimes X_i \pmod{J'}$ for all $1 \leq i \leq n$ with $n \leq \infty$. Note that for all $1 \leq i \leq n$ with $n \leq \infty$, we have $\sum_{j=0}^m (-1)^{m-j} (X_i^{(j)})^o \otimes X_i^{(m-j)} =$

$\xi_i^{(m)} \equiv 0 \pmod{J'}$. Hence, we obtain a series of congruencies modulo J' as follows:

$$\begin{aligned}
(X_i^{(m)})^o \otimes 1 + (-1)^m (1^o \otimes X_i^{(m)}) &\equiv - \sum_{j=1}^{m-1} (-1)^{m-j} (X_i^{(j)})^o \otimes X_i^{(m-j)} \\
&= - \sum_{j=1}^{m-1} (-1)^{m-j} \left((X_i^{(j)})^o \otimes 1 \right) \left(1^o \otimes X_i^{(m-j)} \right) \\
&\equiv - \sum_{j=1}^{m-1} (-1)^{m-j} \left(1^o \otimes X_i^{(j)} \right) \left(1^o \otimes X_i^{(m-j)} \right) \\
&\equiv - \sum_{j=1}^{m-1} (-1)^{m-j} (1^o \otimes X_i^{(j)} X_i^{(m-j)}) \\
&\equiv - \sum_{j=1}^{m-1} (-1)^{m-j} \binom{m}{j} (1^o \otimes X_i^{(m)})
\end{aligned}$$

where the third step uses the inductive hypothesis. The claim now follows from the well-known equality $\sum_{j=1}^{m-1} (-1)^{m-j} \binom{m}{j} = -1 - (-1)^m$.

Now let $\beta \in J \subseteq B^e$ be an arbitrary element. It follows from (6.5.1) that there exists an element $b_\beta \in B$ such that $\beta \equiv 1 \otimes b_\beta \pmod{J'}$. Since $\pi_B(\beta) = 0$, we have $\pi_B(1 \otimes b_\beta) = b_\beta = 0$. Hence, $\beta \equiv 0 \pmod{J'}$, which means that $\beta \in J'$. This implies that $J \subseteq J'$, as desired. \square

Lemma 6.6. *Let $B = A\langle X_1, \dots, X_n \rangle$ with $n \leq \infty$, where A is a divided power DG R -algebra. The set*

$$\text{Mon}(\Omega) = \begin{cases} \{ \xi_1^{(m_1)} \dots \xi_n^{(m_n)} \mid \xi_i^{(m_i)} \in \Omega \ (1 \leq i \leq n) \} \cup \{1^o \otimes 1\} & \text{if } n < \infty \\ \{ \xi_{i_1}^{(m_{i_1})} \dots \xi_{i_t}^{(m_{i_t})} \mid \xi_{i_j}^{(m_{i_j})} \in \Omega \ (i_j \in \mathbb{N}, t < \infty) \} \cup \{1^o \otimes 1\} & \text{if } n = \infty \end{cases}$$

is a basis for the underlying graded free B^o -module B^e . Also, the diagonal ideal J is a free graded B^o -module with the graded basis $\text{Mon}(\Omega) \setminus \{1^o \otimes 1\}$.

Proof. Recall from 2.11 that the underlying graded B^o -module B^e is free with the basis $\text{Mon}(\Gamma)$. We prove the lemma for $n < \infty$; the case of $n = \infty$ is similar.

Assume that $n < \infty$. For an integer $\ell \geq 0$ let

$$\text{Mon}_\ell(\Gamma) = \{ (1^o \otimes X_1^{(m_1)} X_2^{(m_2)} \dots X_n^{(m_n)}) \in \text{Mon}(\Gamma) \mid m_1 + \dots + m_n = \ell \}$$

$$\text{Mon}_\ell(\Omega) = \{ \xi_1^{(m_1)} \xi_2^{(m_2)} \dots \xi_n^{(m_n)} \in \text{Mon}(\Omega) \mid m_1 + \dots + m_n = \ell \}.$$

Also let $F_\ell(B^e)$ be the free B^o -submodule of B^e generated by $\bigcup_{0 \leq i \leq \ell} \text{Mon}_i(\Gamma)$, i.e.,

$$F_\ell(B^e) = \left(\bigcup_{0 \leq i \leq \ell} \text{Mon}_i(\Gamma) \right) B^o = \sum_{i=0}^{\ell} \text{Mon}_i(\Gamma) B^o$$

Then the family $\{F_\ell(B^e) \mid \ell \geq 0\}$ is a filtration of the B^o -module B^e satisfying the following properties:

- (1) $B^o \otimes 1 = F_0(B^e) \subset F_1(B^e) \subset \dots \subset F_\ell(B^e) \subset F_{\ell+1}(B^e) \subset \dots \subset B^e$;
- (2) $\bigcup_{\ell \geq 0} F_\ell(B^e) = B^e$;
- (3) $F_\ell(B^e) F_{\ell'}(B^e) \subseteq F_{\ell+\ell'}(B^e)$; and
- (4) each $F_\ell(B^e)/F_{\ell-1}(B^e)$ is a free B^o -module with free basis $\text{Mon}_\ell(\Gamma)$.

Regarding B^e as a B^o -module, by (6.4.2) for all $1 \leq i \leq n$ and $m \geq 1$ we have

$$\xi_i^{(m)} = (-1)^m (1^o \otimes X_i^{(m)}) + \sum_{j=1}^m (-1)^{m-j} (1^o \otimes X_i^{(m-j)}) (X_i^{(j)})^o.$$

Hence, $\xi_i^{(m)} - (-1)^m (1^o \otimes X_i^{(m)}) \in F_{m-1}(B^e)$. Therefore, if $\xi_1^{(m_1)} \xi_2^{(m_2)} \dots \xi_n^{(m_n)} \in \text{Mon}_\ell(\Omega)$, then we get a sequence of congruencies modulo $F_{\ell-1}(B^e)$ as follows:

$$\begin{aligned} \xi_1^{(m_1)} \xi_2^{(m_2)} \dots \xi_n^{(m_n)} &\equiv (-1)^{m_1} (1^o \otimes X_1^{(m_1)}) \xi_2^{(m_2)} \dots \xi_n^{(m_n)} \\ &\equiv \dots \\ &\equiv (-1)^{m_1 + \dots + m_n} (1^o \otimes X_1^{(m_1)} X_2^{(m_2)} \dots X_n^{(m_n)}) \\ &= (-1)^\ell (1^o \otimes X_1^{(m_1)} X_2^{(m_2)} \dots X_n^{(m_n)}). \end{aligned}$$

Thus, $\text{Mon}_\ell(\Omega)$ is a basis for the B^o -module $F_\ell(B^e)/F_{\ell-1}(B^e)$. By induction on ℓ we can see that $F_\ell(B^e)$ itself is also a free B^o -module with basis $\bigcup_{0 \leq i \leq \ell} \text{Mon}_i(\Omega)$. In particular, every finite subset of $\text{Mon}(\Omega)$ is linearly independent over B^o . Since $\bigcup_{\ell \geq 0} F_\ell(B^e) = B^e$, the set $\text{Mon}(\Omega)$ generates B^e as a B^o -module. Therefore, $\text{Mon}(\Omega)$ is a basis of the free B^o -module B^e .

The fact that the diagonal ideal J is free over B^o with the basis $\text{Mon}(\Omega) \setminus \{1^o \otimes 1\}$ follows from the short exact sequence (3.3.1). \square

Theorem 6.7. *Let $B = A\langle X_1, \dots, X_n \rangle$ with $n \leq \infty$, where A is a divided power DG R -algebra. The DG algebra homomorphism $B^o \rightarrow B^e$ defined by $b^o \mapsto b^o \otimes 1$ is a free extension of DG R -algebras, i.e., $B^e = B^o\langle \xi_1, \xi_2, \dots, \xi_n \rangle$ with $n \leq \infty$.*

Proof. By Lemma 6.6, the set $\text{Mon}(\Omega)$ is a basis for the underlying graded free B^o -module B^e . To complete the proof, it suffices to show that for each $1 \leq i \leq n$ with $n \leq \infty$ the element $d^{B^e}(\xi_i)$ belongs to $B^o\langle \xi_1, \xi_2, \dots, \xi_{i-1} \rangle$ and is a cycle. To see this, note that we have the equalities

$$d^{B^e}(\xi_i) = (d^B(X_i))^o \otimes 1 - 1^o \otimes d^B(X_i) = (d^{A^{(i)}}(X_i))^o \otimes 1 - 1^o \otimes d^{A^{(i)}}(X_i)$$

in which $d^{A^{(i)}}(X_i) \in A^{(i-1)}$ is a cycle. Applying Lemma 6.6 to $A^{(i-1)}$, we see that $(A^{(i-1)})^e$ is generated by $\{\xi_1^{(m_1)} \dots \xi_{i-1}^{(m_{i-1})} \mid m_j \geq 0 \ (1 \leq j \leq i-1)\}$ as an $(A^{(i-1)})^o$ -module. Since $(d^{A^{(i)}}(X_i))^o \otimes 1 - 1^o \otimes d^{A^{(i)}}(X_i) \in (A^{(i-1)})^e$ and $d^{A^{(i)}}(X_i) \in A^{(i-1)}$ is a cycle, $d^{B^e}(\xi_i) \in (A^{(i-1)})^o\langle \xi_1, \dots, \xi_{i-1} \rangle \subseteq B^o\langle \xi_1, \dots, \xi_{i-1} \rangle$ and is a cycle. \square

Our next move is to define the notion of divided powers of the diagonal ideal J .

6.8. Let $B = A\langle X_1, \dots, X_n \rangle$ with $n \leq \infty$, where A is a divided power DG R -algebra. Recall from Lemma 6.5 that $J = \Omega B^e$. For an integer $\ell \geq 0$ let

$$\text{Mon}_{\geq \ell}(\Omega) = \begin{cases} \{\xi_1^{(m_1)} \dots \xi_n^{(m_n)} \in \text{Mon}(\Omega) \mid m_1 + \dots + m_n \geq \ell\} & \text{if } n < \infty \\ \{\xi_{i_1}^{(m_{i_1})} \dots \xi_{i_t}^{(m_{i_t})} \in \text{Mon}(\Omega) \mid m_{i_1} + \dots + m_{i_t} \geq \ell\} & \text{if } n = \infty. \end{cases}$$

which is the set of monomials that are (symbolically) products of more than or equal to ℓ variables. By Lemma 6.6, the set $\text{Mon}_{\geq 1}(\Omega) = \text{Mon}(\Omega) \setminus \{1^o \otimes 1\}$ is a basis for J as a free B^o -module. We define the ℓ -th power of J to be $J^{(\ell)} := \text{Mon}_{\geq \ell}(\Omega) B^e$. Note that $J = J^{(1)}$ and there is a descending sequence

$$B^e \supset J \supset J^{(2)} \supset \dots \supset J^{(\ell)} \supset J^{(\ell+1)} \supset \dots$$

of the DG ideals in B^e ; see Lemma 6.9 below.

Lemma 6.9. *Let $B = A\langle X_1, \dots, X_n \rangle$ with $n \leq \infty$, where A is a divided power DG R -algebra. For every $\ell \geq 0$ the ideal $J^{(\ell)}$ is a DG ideal of B^e and we have $JJ^{(\ell)} = J^{(\ell)}J \subseteq J^{(\ell+1)}$. Moreover, the quotient $J^{(\ell)}/J^{(\ell+1)}$ is a DG B -module.*

Proof. We prove the assertion for $n < \infty$. The case where $n = \infty$ is treated similarly by using the appropriate notation.

By Theorem 6.7, for all $1 \leq i \leq n$ and $m_1, \dots, m_n \geq 0$ we have

$$\xi_i^{(m_1)} \xi_2^{(m_2)} \dots \xi_n^{(m_n)} = (-1)^{|\xi_i|(\sum_{j=1}^{i-1} m_j |\xi_j|)} (m_i + 1) \xi_1^{(m_1)} \dots \xi_i^{(m_i+1)} \dots \xi_n^{(m_n)}.$$

This shows that $JJ^{(\ell)} \subseteq J^{(\ell+1)}$. Similarly, $J^{(\ell)}J \subseteq J^{(\ell+1)}$.

To prove that $J^{(\ell)}$ is a DG ideal, we show that $d^{B^e}(J^{(\ell)}) \subseteq J^{(\ell)}$. Recall, from the definition, that J is the kernel of the DG R -algebra homomorphism π_B . Since π_B is a chain map, we have $\pi_B(d^{B^e}(J)) = d^B(\pi_B(J)) = 0$. Hence, $d^{B^e}(J) \subseteq J$.

Note that $d^{B^e}(\xi_i^{(m)}) \in J^{(m)}$ for all $1 \leq i \leq n$ and $m \geq 1$. In fact, we have $d^{B^e}(\xi_i^{(m)}) = \xi_i^{(m-1)} d^{B^e}(\xi_i) \in J^{(m-1)}J \subseteq J^{(m)}$.

Now assume that $\ell \geq 2$. If $m_1 + \dots + m_n \geq \ell$, then we have

$$d^{B^e}(\xi_1^{(m_1)} \xi_2^{(m_2)} \dots \xi_n^{(m_n)}) = \sum_{i=1}^n \pm d^{B^e}(\xi_i^{(m_i)}) \left(\xi_1^{(m_1)} \dots \xi_{i-1}^{(m_{i-1})} \xi_{i+1}^{(m_{i+1})} \dots \xi_n^{(m_n)} \right)$$

which is an element in $\sum_{i=1}^n J^{(m_i)} J^{(\ell-m_i)} \subseteq J^{(\ell)}$. Therefore, $d^{B^e}(J^{(\ell)}) \subseteq J^{(\ell)}$.

The assertion that $J^{(\ell)}/J^{(\ell+1)}$ is a DG B -module follows from the facts that the underlying graded B^e -module $J^{(\ell)}/J^{(\ell+1)}$ is annihilated by J and $B^e/J \cong B$ as graded algebras. \square

Theorem 6.10. *Let $B = A\langle X_1, \dots, X_n \rangle$ with $n \leq \infty$, where A is a divided power DG R -algebra. For every $\ell \geq 0$, the DG B -module $J^{(\ell)}/J^{(\ell+1)}$ is semifree with the semifree basis $\text{Mon}_\ell(\Omega)$. In case that $n < \infty$, this is a finite semifree basis.*

Proof. Recall from 2.10 that B^o is a DG R -subalgebra of B^e . By definition of $J^{(\ell)}$ from 6.8, the underlying graded B^o -module $J^{(\ell)}/J^{(\ell+1)}$ is free with the basis $\text{Mon}_\ell(\Omega)$. Note that the composition of the maps $B^o \rightarrow B^e \xrightarrow{\pi_B} B$ defined by $b^o \mapsto b^o \otimes 1 \mapsto b$ is an isomorphism, and that the B -module structure on $J^{(\ell)}/J^{(\ell+1)}$ from Lemma 6.9 coincides with its B^o -module structure. Thus, $J^{(\ell)}/J^{(\ell+1)}$ is free as an underlying graded B -module. Therefore, $J^{(\ell)}/J^{(\ell+1)}$ is a semifree DG B -module with semifree basis $\text{Mon}_\ell(\Omega)$. \square

6.11. Let $B = A[X_1, \dots, X_n]$ with $n \leq \infty$ be a polynomial extension of the DG R -algebra A with variables X_1, \dots, X_n of positive degrees. For each $1 \leq i \leq n$, consider the diagonal ξ_i of the variable X_i defined in (6.4.1). In this case we have

$$\xi_i^m = \sum_{j=0}^m (-1)^{m-j} \binom{m}{j} \left((X_i^j)^o \otimes X_i^{m-j} \right). \quad (6.11.1)$$

Hence, similar to 6.4, we can consider the set $\{\xi_i^m \mid 1 \leq i \leq n, m \in \mathbb{N}\} \subseteq J$, which we again denote by Ω in this case. Replacing divided powers $X_i^{(m)}$ and $\xi_i^{(m)}$ by ordinary powers X_i^m and ξ_i^m , we can show that Lemmas 6.5 and 6.6 hold in this case as well. Hence, similar to Theorem 6.7, we have $B^e = B^o[\xi_1, \dots, \xi_n]$. Note that in this case for an integer $\ell \geq 0$ we have $J^\ell = \text{Mon}_{\geq \ell}(\Omega)B^e$ and $J^\ell/J^{\ell+1}$ is a semifree DG B -module with the semifree basis $\text{Mon}_\ell(\Omega)$.

We can now prove the following which is a key to the proof of Main Theorem.

Corollary 6.12. *Let $n \leq \infty$. We consider the following two cases:*

- (a) $B = A[X_1, \dots, X_n]$; or
- (b) A is a divided power DG R -algebra and $B = A\langle X_1, \dots, X_n \rangle$.

Then B is DG quasi-smooth over A . If $n < \infty$, then B is DG smooth over A .

Proof. Note that by definition of $J^{(\ell)}$ and Lemma 6.6, the quotient $J/J^{(\ell)}$ is a semifree DG B -module with the semifree basis $\text{Mon}(\Omega) \setminus \text{Mon}_{\geq \ell}(\Omega)$. In case (a), set $J^{[\ell]} = J^{(\ell)}$ and in case (b), set $J^{[\ell]} = J^\ell$ for each positive integer ℓ . The assertion follows from Lemma 6.9, Theorem 6.10, and 6.11. \square

6.13. *Proof of Main Theorem.* The assertion follows from Theorem 1.2 and Corollary 6.12. \square

The following result follows from Main Theorem(a) and 2.3.

Corollary 6.14. *Assume that $A = R$, or A is a DG R -algebra with R containing the field of rational numbers, and let $B = A\langle X_1, \dots, X_n \rangle$. If N is a bounded below semifree DG B -module such that $\text{Ext}_B^i(N, N) = 0$ for all $i \geq 1$, then N is naïvely liftable to A . Moreover, N is a direct sum of a DG B -module that is liftable to A .*

7. AUSLANDER-REITEN CONJECTURE AND NAÏVE LIFTING PROPERTY

Our study in this paper is motivated by the following long-standing conjecture posed by Auslander and Reiten which has been studied in numerous works; see for instance [1, 2, 5, 6, 10, 14, 15, 16, 24, 25, 26, 29, 30], to name a few.

Auslander-Reiten Conjecture ([3, p. 70]). *Let (S, \mathfrak{n}) be a local ring and M be a finitely generated S -module. If $\text{Ext}_S^i(M \oplus S, M \oplus S) = 0$ for all $i > 0$, then M is a free S -module.*

Our Main Theorem in this paper considers naïve liftability of DG modules along *finite* free extensions of DG algebras. However, in dealing with the Auslander-Reiten Conjecture, we need to work with *infinite* free extensions of DG algebras. So, we pose the following conjecture for which we do not have a proof yet.

Naïve Lifting Conjecture. *Assume that A is a divided power DG R -algebra, and let $B = A\langle X_i \mid i \in \mathbb{N} \rangle$. If N is a bounded below semifree DG B -module such that $\text{Ext}_B^i(N \oplus B, N \oplus B) = 0$ for all $i \geq 1$, then N is naïvely liftable to A .*

Our next result explains the relation between these conjectures.

Theorem 7.1. *If Naïve Lifting Conjecture holds, then the Auslander-Reiten Conjecture holds.*

Proof. Let (S, \mathfrak{n}) be a local ring and M be a finitely generated S -module with $\text{Ext}_S^i(M \oplus S, M \oplus S) = 0$ for all $i > 0$. Without loss of generality we can assume that S is complete in its \mathfrak{n} -adic topology. Consider the minimal Cohen presentation $S \cong R/I$ of S , where R is a regular local ring and I is an ideal of R . By a construction of Tate [32], there is a DG R -algebra $B = R\langle X_i \mid i \in \mathbb{N} \rangle$ that resolves S as an R -module, that is, $S \simeq B$. The S -module M is regarded as a DG B -module via the natural augmentation $B \rightarrow S$. This homomorphism of DG S -algebras induces a functor $\mathfrak{F}: \mathcal{D}(S) \rightarrow \mathcal{D}(B)$ of the derived categories. Since $S \simeq B$, by Keller's Rickard Theorem [19], the functor \mathfrak{F} yields a triangle equivalence and its quasi-inverse is given by $-\otimes_B^L S$.

Let $N \xrightarrow{\cong} M$ be a semifree resolution of the DG B -module M ; see [4] for more information. Then, as an underlying graded free B -module, N is non-negatively graded and $H(N) \cong H(M) = M$, which is bounded and finitely generated over $H_0(B) \cong R$. Note that M corresponds to N and S corresponds to B under the functor \mathfrak{F} . Since \mathfrak{F} is a triangle equivalence, we conclude that $\text{Ext}_B^i(N \oplus B, N \oplus B) = 0$ for all $i \geq 1$. By our assumption, N is naively liftable to A . In particular, by Corollary 5.5, N is a direct summand of $N|_R \otimes_R B$. Using the category equivalence \mathfrak{F} , we see that M is a direct summand of $M \otimes_R^L S$ in $\mathcal{D}(S)$ which is a bounded free complex over S , since R is regular. Hence, $\text{pd}_S(M) < \infty$. It then follows from [11, Theorem 2.3] that M is free over S . \square

7.2. According to the proof of Theorem 7.1, we do not need to prove the Naïve Lifting Conjecture in its full generality for the Auslander-Reiten Conjecture; only proving it for the case where $A = R$ is a regular local ring would suffice for this purpose. Note that, despite the finite free extension case, the assumption “ $\text{Ext}_B^i(N, N) = 0$ for all $i \geq 1$ ” is not enough for the Naïve Lifting Conjecture to be true in general. The reason is that there exist non-free finitely generated modules M over a general local ring S satisfying $\text{Ext}_S^i(M, M) = 0$ for all $i \geq 1$; see, for instance, [17].

7.3. In the proof of Theorem 7.1, if S is resolved as an R -module by a finite free extension $B = R\langle X_1, \dots, X_n \rangle$, then S is known to be a complete intersection ring. Hence, in this case, our Main Theorem and Theorem 7.1 just provide another proof for the well-known fact that complete intersection rings satisfy the Auslander-Reiten Conjecture; see, for instance, [1, 5, 16].

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DEPARTMENT OF MATHEMATICAL SCIENCES, GEORGIA SOUTHERN UNIVERSITY, STATESBORO, GA 30460, U.S.A.

Email address: snasseh@georgiasouthern.edu

INSTITUTE FOR THE ADVANCEMENT OF HIGHER EDUCATION, OKAYAMA UNIVERSITY OF SCIENCE, RIDAICHO, KITAKU, OKAYAMA 700-0005, JAPAN

Email address: ono@pub.ous.ac.jp

GRADUATE SCHOOL OF NATURAL SCIENCE AND TECHNOLOGY, OKAYAMA UNIVERSITY, OKAYAMA 700-8530, JAPAN

Email address: yoshino@math.okayama-u.ac.jp