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DEGENERATIONS OVER (A_∞)-SINGULARITIES AND CONSTRUCTION OF DEGENERATIONS OVER COMMUTATIVE RINGS

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Abstract. We give a necessary condition of degeneration via matrix representations, and consider degenerations of indecomposable Cohen-Macaulay modules over hypersurface singularities of type (A_∞). We also provide a method to construct degenerations of finitely generated modules over commutative rings.

1. Introduction

The notion of degenerations of modules has been a central subject in the representation theory of algebras. Let k be a field, and let R be a k-algebra. Let M, N be R-modules with the same finite dimension as k-vector spaces. We say that M *degenerates to* N if in the module variety the point corresponding to M belongs to the Zariski closure of the orbit of the point corresponding to N. This definition is only available for modules of finite length. The third author [9] thus introduces a scheme-theoretic definition of degenerations for arbitrary finitely generated modules, and extends the characterization of degenerations due to Riedtmann [5] and Zwara [12]. Interest of this subject is to describe relations of degenerations of modules [2, 5, 8, 9, 11, 12]. In [3], the first and third authors give a complete description of degenerations over a ring of even-dimensional simple hypersurface singularity of type (A_n).

The first purpose of this paper is to give a necessary condition for the degenerations of maximal Cohen-Macaulay modules by considering it via matrix representations over the base regular local ring. As an application, we give the description of degenerations of indecomposable Cohen-Macaulay modules over hypersurface singularities of type (A_∞).

Theorem 1.1. *Let k be an algebraically closed field.*

(1) *Let $R = k[\![x, y]\!]/(x^2)$ be a hypersurface of dimension one. Then $(x, y^a)R$ degenerates to $(x, y^b)R$ if and only if $a \le b$ and $a \equiv b \mod 2$.*
(2) *Let $R = k[\![x, y, z]\!]/(xy)$ be a hypersurface of dimension two. Then for all $a < b$, $(x, z^a)R$ and $(y, z^a)R$ do not degenerate to $(x, z^b)R$ and $(y, z^b)R$, respectively.*

The proof of this theorem requires the fact that the Bass-Quillen conjecture holds for a formal power series ring over a field ([4, Chapter V Theorem 5.1]), which is the only place where the assumption that R is complete and equicharacteristic is necessary. Our proof also requires the base field to be algebraically closed. Note that a complete equicharacteristic local hypersurface R has countable Cohen-Macaulay representation type if and only if R has either of the two forms in Theorem 1.1.

The second purpose of this paper is to provide a method to construct degenerations. We obtain the following theorem.

Theorem 1.2. *Let R be a commutative noetherian algebra over a field, and let L be a finitely generated R-module. Let α be an endomorphism of L with $\operatorname{Im}\alpha = \operatorname{Ker}\alpha$, and let $x \in R$ be an L-regular element. Then the following statements hold.*

(1) *Any submodule M of L containing $\alpha(L) + xL$ degenerates to $N := \alpha(M) + x^2L$.*
(2) *If L, M are Cohen-Macaulay, then so is N.*

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This theorem is actually a corollary of our main result in this direction, which holds over arbitrary commutative rings and is proved by elementary calculations of matrices; we do not use matrix representations. We also give various applications, one of which gives an alternative proof of the 'if' part of Theorem 1.1(1).

The organization of this paper is as follows. In Section 2, we review the notion of degenerations of Cohen-Macaulay modules by means of matrix representations, and give a necessary condition for degenerations (Corollary 2.8). Using this, in Section 3, we describe degenerations over hypersurface singularities of type (A_∞) with dimension at most two (Theorems 3.1 and 3.2). In Section 4, we give several ways to construct degenerations (Theorem 4.2), and provide applications. In Section 5 we attempt to extend Theorem 1.1 to higher dimension (Example 5.5).

2. Matrix representation

We begin with recalling the definition of degenerations of finitely generated modules. For details, we refer the reader to [8, 9, 10].

Definition 2.1. Let R be a noetherian algebra over a field k. Let M, N be finitely generated left R-modules. We say that M *degenerates to* N (or N *is a degeneration of* M) if there exists a discrete valuation k-algebra (V, tV, k) and a finitely generated left $R \otimes_k V$-module Q such that Q is flat as a V-module, $Q/tQ \cong N$ as an R-module, and $Q_t \cong M \otimes_k V_t$ as an $R \otimes_k V_t$-module. When this is the case, we say that M *degenerates to* N *along* V.

Remark 2.2. Let R be a noetherian algebra over a field k and let M, N and L be finitely generated left R-modules.
(1) M degenerates to N if and only if there exists a short exact sequence of finitely generated left R-modules

$$0 \longrightarrow Z \xrightarrow{\binom{h}{g}} M \oplus Z \longrightarrow N \longrightarrow 0,$$

where the endomorphism g of Z is nilpotent. See [12, Theorem 1] and [9, Theorem 2.2]. Note that if M and N is Cohen-Macaulay, then so is Z (see [9, Remark 4.3]).
(2) Suppose that there is an exact sequence $0 \to L \to M \to N \to 0$. Then M degenerates to $L \oplus N$. See [9, Remark 2.5] for the detail.
(3) Assume that R is commutative and suppose that M degenerates to N. Then the ith Fitting ideal of M contains that of N for all $i \geq 0$. Namely, denoting the ith Fitting ideal of an R-module M by $\mathcal{F}_i^R(M)$, we have $\mathcal{F}_i^R(M) \supseteq \mathcal{F}_i^R(N)$ for all $i \geqq 0$. See [10, Theorem 2.5].

Remark 2.3. We can always take the localization $k[t]_{(t)}$ of a polynomial ring as a discrete valuation ring V as in Definition 2.1. Moreover, let $T = k[t] \backslash (t)$ and $T' = k[t] \backslash \{0\}$. Then we also have $R \otimes_k V = T^{-1}R[t]$ and $R \otimes_k V_t = T'^{-1}R[t]$. See [9, Corollary 2.4].

Now let us recall the definition of a matrix representation.

Definition 2.4. Let R be a commutative noetherian local ring. Let S be a Noether normalization of R, that is, S is a subring of R which is a regular local ring such that R is a finitely generated S-module. Then R is Cohen-Macaulay if and only if R is S-free, and a finitely generated R-module is (maximal) Cohen-Macaulay[1] if and only if M is S-free.

Let R, M be Cohen-Macaulay. Then $M \cong S^n$ for some $n \geq 0$, and we have a k-algebra homomorphism

$$R \to \mathrm{End}_S(M) \cong S^{n \times n}.$$

This is called a *matrix representation* of M over S.

Example 2.5. (1) Let S be a regular local ring and $R = S[\![x]\!]/(f)$ where $f = x^n + a_1 x^{n-1} + \cdots + a_n$ with $a_i \in S$. Then R is a Cohen-Macaulay ring with an S-basis $\{1, x, x^2, ..., x^{n-1}\}$ and each matrix representation of Cohen-Macaulay R-module M comes from the action of x on M.
(2) Let $R = k[\![x, y]\!]/(x^2)$ with a field k of characteristic not two. It is known that the isomorphism classes of indecomposable Cohen-Macaulay R-modules are the following:

$$R, \quad R/(x), \quad (x, y^n) \quad n \geq 1.$$

[1]Throughout this paper, "Cohen-Macaulay" means "maximal Cohen-Macaulay".

Giving the matrix representations over $k[\![y]\!]$ of these modules is equivalent to giving the following square-zero matrices; see also [7, (6.5)].

$$\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \quad (0), \quad \left(\begin{smallmatrix} 0 & y^n \\ 0 & 0 \end{smallmatrix}\right) \quad n \geq 1.$$

Proposition 2.6. *Let $(R, \mathfrak{m}, k)$ be a commutative noetherian complete equicharacteristic local ring. Then one can take a formal power series ring S over k as a Noether normalization of R. Assume that k is algebraically closed. Let M, N be Cohen-Macaulay R-modules such that M degenerates to N along V. Let Q be a finitely generated $R \otimes_k V$-module which gives the degeneration. Then Q is free as an $S \otimes_k V$-module.*

Proof. Since $V = T^{-1}k[t]$ where $T = k[t] \backslash (t)$ by Remark 2.3, we can take a finitely generated $R[t]$-submodule Q' of Q such that $T^{-1}Q' = Q$. Then Q' is flat over $k[t]$, $Q'_0 \cong N$ and $Q'_c \cong M$ for each $c \in k^\times$. Here Q'_c is defined to be $Q'/(t-c)Q'$ for an element $c \in k$; see [9, Theorem 3.2]. We show that Q' is free as an $S[t]$-module. Set $T' = k[t] \backslash \{0\}$ and $\mathfrak{p}$ be a prime ideal of $S[t]$.

Suppose $T' \cap \mathfrak{p} \neq \emptyset$. Since k is an algebraically closed field, there is $c \in k$ such that $t - c \in \mathfrak{p}$. Then,

$$Q'_{\mathfrak{p}}/(t-c)Q'_{\mathfrak{p}} \cong (Q'/(t-c)Q')_{\overline{\mathfrak{p}}}.$$

By the choice of Q', the right-hand side is isomorphic to $N_{\overline{\mathfrak{p}}}$ or $M_{\overline{\mathfrak{p}}}$, so that $S_{\overline{\mathfrak{p}}}$-free. Thus $Q'_{\mathfrak{p}}/(t-c)Q'_{\mathfrak{p}}$ is free as an $S_{\overline{\mathfrak{p}}} = (S[t]/(t-c))_{\overline{\mathfrak{p}}}$-module. Thus, $Q'_{\mathfrak{p}}$ is a Cohen-Macaulay module over the regular local ring $S[t]_{\mathfrak{p}}$. Hence, $Q'_{\mathfrak{p}}$ is free as an $S[t]_{\mathfrak{p}}$-module.

Suppose $T' \cap \mathfrak{p} = \emptyset$. Then

$$Q'_{\mathfrak{p}} \cong (T'^{-1}Q')_{\mathfrak{p}'} \cong (T'^{-1}(T^{-1}Q')_{\mathfrak{p}'} \cong (T'^{-1}Q)_{\mathfrak{p}'} \cong (M \otimes_k V_t)_{\mathfrak{p}'}.$$

Hence $Q'_{\mathfrak{p}}$ is a free $S[t]_{\mathfrak{p}}$-module.

Therefore Q' is a projective $S[t]$-module. By the fact on the Bass-Quillen conjecture [4, Chapter V Theorem 5.1], each projective $S[t]$-module is $S[t]$-free since S is a formal power series ring. Hence Q' is free as an $S[t]$-module, so that $Q = T^{-1}Q'$ is free as a $T^{-1}S[t] = S \otimes_k V$-module. □

The above proposition enables us to consider the matrix representation of Q over $S \otimes_k V$ because it is free. For matrix representations $\mu, \nu : R \to S^{n \times n}$, we write $\mu \cong \nu$ if there exists an invertible matrix α such that $\alpha^{-1}\mu(r)\alpha = \nu(r)$ for each $r \in R$. Such a matrix α is called an *interwining matrix* of μ and ν.

Corollary 2.7. *Let R, k, S, V be as in Proposition 2.6. Let M and N be Cohen-Macaulay R-modules and μ and ν the matrix representations of M and N over S respectively. Then M degenerates to N along V if and only if there exists a matrix representation ξ over $S \otimes_k V$ such that $\xi \otimes_V V/tV \cong \nu$ and $\xi \otimes_V V_t \cong \mu \otimes_k V_t$ hold.*

Under the situation of the proposition above, R is a finitely generated S-module. Set $u_1, ..., u_t$ be generators of R as an S-module. Then giving a matrix representation μ over S is equivalent to giving a tuple of matrices $(\mu_1, ..., \mu_t)$, where $\mu_i = \mu(u_i)$ for $i = 1, ..., t$.

Corollary 2.8. *Let R, k, S, V be as in Proposition 2.6 and let M be a Cohen-Macaulay R-module. Let Q be a free $S \otimes_k V$-module and suppose that Q_t is isomorphic to $M \otimes_k V_t$. We denote by $(\mu_1, \ldots, \mu_t)$ (resp. $(\xi_1, \ldots, \xi_t)$) the matrix representation of M (resp. Q) over S (resp. $S \otimes_k V$). Then we have the following equalities in $S \otimes_k V$ for $i = 1, \ldots, t$:*

(1) $\mathrm{tr}(\xi_i) = \mathrm{tr}(\mu_i)$ *and* $\det(\xi_i) = \det(\mu_i)$,
(2) *For all $j \geq 0$, there exist l, l' such that $I_j(\xi_i) = t^l I_j(\mu_i)$ and $t^{l'} I_j(\xi_i) = I_j(\mu_i)$.*

Here, $I_j(-)$ stands for the ideal generated by the j-minors.

Proof. By the assumption, there exists an invertible matrix α with entries in $S \otimes_k V_t$ such that $\alpha^{-1}\xi_i\alpha = \mu_i$. Thus $\mathrm{tr}(\xi_i) = \mathrm{tr}(\mu_i)$ and $\det(\xi_i) = \det(\mu_i)$. Moreover we have $I_j(\xi_i) = I_j(\mu_i)$ in $S \otimes_k V_t$. Hence we have the equalities in the lemma. □

Taking Corollary 2.7 into account, Corollary 2.8 gives a necessary condition of the degeneration. In the next section, we apply the condition to (A^d_∞)-singularities of dimension at most two.

3. Degenerations of Cohen-Macaulay modules over (A_∞)-singularities

Recall that a Cohen-Macaulay local ring R is said to have *countable Cohen-Macaulay representation type* if there exist infinitely but only countably many isomorphism classes of indecomposable Cohen-Macaulay R-modules. Let R be a complete equicharacteristic local hypersurface with residue field k of characteristic not two. Then R has countable Cohen-Macaulay representation type if and only if R is isomorphic to the ring $k[\![x_0, x_1, x_2, \cdots, x_d]\!]/(f)$, where f is either of the following:

$$f = \begin{cases} x_1^2 + \cdots + x_d^2 & (A_\infty), \\ x_0^2 x_1 + x_2^2 + \cdots + x_d^2 & (D_\infty). \end{cases}$$

Moreover, when this is the case, the indecomposable Cohen-Macaulay R-modules are classified completely; we refer the reader to [1, §1] for more information.

In this section, we describe the degenerations of indecomposable Cohen-Macaulay R-modules in the case where R has type (A_∞) and $\dim R = 1, 2$. Throughout the rest of this section, we assume that k is an algebraically closed field of characteristic not two. Let us start by the 1-dimensional case.

Theorem 3.1. *Let $R = k[\![x, y]\!]/(x^2)$. Then (x, y^a) degenerates to (x, y^b) if and only if $a \leq b$ and $a \equiv b \mod 2$.*

Proof. First we notice that $a \leq b$ if (x, y^a) degenerates to (x, y^b) by Remark 2.2(3). As mentioned in Example 2.5(2), the matrix representations of (x, y^n) for $n \geq 0$ are $\left(\begin{smallmatrix} 0 & y^n \\ 0 & 0 \end{smallmatrix}\right)$. Here $(x, y^0) = R$. Suppose that $a \equiv b \mod 2$. We consider $R \otimes_k V$-module Q whose matrix representation ξ over $S \otimes_k V$ is

$$\xi = \begin{pmatrix} ty^{\frac{a+b}{2}} & y^b \\ -t^2 y^a & -ty^{\frac{a+b}{2}} \end{pmatrix}.$$

Then one can show that Q gives the degeneration.

To show the converse, we prove the following claim.

Claim. $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ never degenerates to $\left(\begin{smallmatrix} 0 & y^{2m+1} \\ 0 & 0 \end{smallmatrix}\right)$, that is, R never degenerates to (x, y^{2m+1}).

Proof of Claim. After applying elementary row and column operations, we may assume that the matrix representation of Q over $S \otimes_k V$ which gives the degeneration is of the form:

$$\xi = \begin{pmatrix} 0 & y^{2m+1} \\ 0 & 0 \end{pmatrix} + t \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} t\alpha & y^{2m+1} + t\beta \\ t\gamma & t\delta \end{pmatrix}.$$

Then $\xi \otimes V_t \cong \mu \otimes V_t$. By Corollary 2.8, we have $t\delta = -t\alpha$ since $\mathrm{tr}(\xi) = tr(\mu) = 0$. Moreover,

$$\det \xi = -t^2\alpha^2 - t\gamma(y^{2m+1} + t\beta) = 0, \tag{3.1}$$

$$I_1(\xi) = (t\alpha, t\gamma, y^{2m+1} + t\beta) \supseteq (t^l) \text{ for some } l.$$

Note that the above equalities are obtained in $S \otimes_k V$. From the equation (3.1), we have $t\alpha^2 = \gamma(y^{2m+1} + t\beta)$. Since t does not divide $y^{2m+1} + t\beta$, t divides γ, so that $\gamma = t\gamma'$ for some $\gamma' \in S \otimes_k V$. Hence we also have the equality in $S \otimes_k V$:

$$\alpha^2 = \gamma'(y^{2m+1} + t\beta). \tag{3.2}$$

Since S is factorial, so is $S[t]$. Thus $S \otimes_k V = T^{-1}S[t]$ is also factorial. Take the unique factorization into prime elements of $y^{2m+1} + t\beta$:

$$y^{2m+1} + t\beta = P_1^{e_1} P_2^{e_2} \cdots P_n^{e_n}.$$

Then, there exists i such that e_i is an odd number. Since the equation (3.2) holds, P_i divides α, so that P_i also divides γ'. Therefore, $(P_i) \supseteq I_i(\xi) \supseteq (t^l)$, so that $P_i = t$. This makes contradiction since t cannot divide $y^{2m+1} + t\beta$. ♢

Now we assume that $\left(\begin{smallmatrix} 0 & y^a \\ 0 & 0 \end{smallmatrix}\right)$ degenerates to $\left(\begin{smallmatrix} 0 & y^b \\ 0 & 0 \end{smallmatrix}\right)$. Then we also have the following matrix representation of Q over $S \otimes_k V$:

$$\xi = \begin{pmatrix} 0 & y^b \\ 0 & 0 \end{pmatrix} + t \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} = \begin{pmatrix} t\alpha & y^b + t\beta \\ t\gamma & -t\alpha \end{pmatrix}.$$

Then we also have $t^2\alpha^2 = t\gamma(y^b + t\beta)$, $(t\alpha, t\gamma, y^b + t\beta) \supseteq t^l(y^a)$ for some l, and

$$t^{l'}(t\alpha, t\gamma, y^b + t\beta) \subseteq (y^a) \text{ for some } l'. \tag{3.3}$$

From (3.3), we see that y^a divides $t\alpha$, $t\gamma$ and $y^b + t\beta$. This yields that

$$t^2{\alpha'}^2 = t\gamma'(y^{b-a} + t\beta') \text{ and } (t\alpha', t\gamma', y^{b-a} + t\beta') \supseteq (t^l) \text{ for some } l,$$

where $\alpha = y^a\alpha'$, $\gamma = y^a\gamma'$and $y^b + t\beta = y^a(y^{b-a} + t\beta')$. If $a \not\equiv b \mod 2$, $b - a$ is an odd number. As in the proof of the claim above, this never happen. Therefore $a \equiv b \mod 2$. □

Next we consider the 2-dimensional case, that is, let $R = k[\![x, y, z]\!]/(xy)$. The nonisomorphic indecomposable Cohen-Macaulay R-modules are R and the following ideals; see [1, Proposition 2.2].

$$(x), \quad (y), \quad (x, z^n), \quad (y, z^n) \quad n \geq 1.$$

Theorem 3.2. *Let $R = k[\![x, y, z]\!]/(xy)$. Then (x, z^a) (resp. (y, z^a)) never degenerates to (x, z^b) and (y, z^b) for all $a < b$.*

Proof. Replacing Y with $x - y$, we may consider $k[\![x, Y, z]\!]/(x^2 - Yx)$ as R. We consider the ideals (x, z^a) and $(x - Y, z^a)$ instead of the ones in the assertion. We rewrite Y by y. The matrix representations of (x, z^a) and $(x - y, z^a)$ over $S = k[\![y, z]\!]$ are $\left(\begin{smallmatrix} y & z^a \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & z^a \\ 0 & y \end{smallmatrix}\right)$ respectively. See Remark 3.3 below.

Suppose that $\mu = \left(\begin{smallmatrix} y & z^a \\ 0 & 0 \end{smallmatrix}\right)$ degenerates to $\left(\begin{smallmatrix} y & z^b \\ 0 & 0 \end{smallmatrix}\right)$ along V, and let Q be a finitely generated $R \otimes_k V$-module which gives the degeneration. By elementary row and column operations, we may assume that the matrix representation ξ of Q over $S \otimes_k V$ which gives the degeneration is of the form:

$$\xi = \left(\begin{smallmatrix} y & z^b \\ 0 & 0 \end{smallmatrix}\right) + t\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) = \left(\begin{smallmatrix} y+t\alpha & z^b+t\beta \\ t\gamma & t\delta \end{smallmatrix}\right).$$

Since $\xi \otimes V_t \cong \mu \otimes V_t$, by Corollary 2.8, we have

$$\operatorname{tr}(\xi) = y, \tag{3.4}$$

$$\det(\xi) = (y + t\alpha)t\delta - t\gamma(z^b + t\beta) = 0, \tag{3.5}$$

$$I_1(\xi) = (y + t\alpha, t\gamma, z^b + t\beta, t\delta) \supseteq t^l(z^a, y) \text{ for some } l. \tag{3.6}$$

By (3.4) and (3.5), we have

$$-\alpha(y + t\alpha) = \gamma(z^b + t\beta). \tag{3.7}$$

Set $P = z^b + t\beta$. Then, since $P \equiv z^b$ modulo t, P divides α. Consider the equation (3.7) in $S \otimes_k V/y(S \otimes_k V)$:

$$-t\overline{\alpha} \cdot \overline{\alpha} = \overline{\gamma} \cdot \overline{z^b + t\beta}.$$

Since P divides α, $\overline{P}$ divides $\overline{\alpha}$. Thus $\overline{P}$ also divides $\overline{\gamma}$. This yields that $I_1(\xi) = (y+t\alpha, t\gamma, z^b+t\beta, -t\alpha) \subseteq (P, y)$. By (3.6), $t^l(z^a, y) \subseteq (P, y)$. Let $\mathfrak{p}$ be a minimal prime ideal of (P, y). Notice that $\mathfrak{p}$ has height 2 and contains $t^l z$ and y. Suppose that $\mathfrak{p}$ contains t. Then $\mathfrak{p}$ also contains z since $\mathfrak{p}$ contains $P = z^b + t\beta$, so that $\mathfrak{p} = (z, y, t)$. This makes contradiction. Hence $\mathfrak{p} = (z, y)$. Consider the above inclusion of ideals in $S \otimes_k V/y(S \otimes_k V)$ again, and then we see that $\overline{(P, y)} = (\overline{P}) \ni \overline{t^l z^a}$. Since $(\overline{P})$ are also contained in $\overline{\mathfrak{p}} = (\overline{z})$, $\overline{P} = \overline{P'z}$ for some $\overline{P'} \in S \otimes_k V/y(S \otimes_k V)$. Thus $(\overline{P}) = (\overline{P'z}) \ni \overline{t^l z^a}$, which implies that $(\overline{P'}) \ni \overline{t^l z^{a-1}}$ as an ideal in $S \otimes_k V/y(S \otimes_k V)$. Note that $P' = z^{b-1} + tP''$ in $S \otimes_k V$ and the minimal prime ideal of $(\overline{P'})$ in $S \otimes_k V/y(S \otimes_k V)$ is also $(\overline{z})$. Hence we can repeat the procedure and we obtain the inclusion of ideals in $S \otimes_k V/y(S \otimes_k V)$:

$$(\overline{z^{b-a} + tP'''}) \supseteq (\overline{t^l}).$$

However this never happen if $b - a > 0$. Therefore we obtain the conclusion. The same proof is valid for showing the remaining claim that (x, z^a) (resp. (y, z^a)) does not degenerate to (y, z^b) (resp. (x, z^b) or (y, z^b)). □

Remark 3.3. Let $R = k[\![x, y, z]\!]/(x^2 - xy)$ and let $M = (x, z^n)$, $N = (x - y, z^n)$ be ideals. Set $S = k[\![y, z]\!]$. Note that the matrix representations of M and N over S are obtained from the action of x on M and N respectively. Since M has a basis x and z^n as an S-module, that is, $M \cong xS \oplus z^nS$, the multiplication map a_x by x on M induces the correspondence:

$$a_x : xS \oplus z^nS \to xS \oplus z^nS; \quad \left(\begin{smallmatrix} ax \\ bz^n \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} ayx+bz^nx \\ 0 \end{smallmatrix}\right).$$

Hence the matrix representation of M over S is $\left(\begin{smallmatrix} y & z^n \\ 0 & 0 \end{smallmatrix}\right)$. Similarly, since $N \cong (x-y)S \oplus z^nS$ as an S-module, the multiplication map b_x on N induces

$$b_x : (x-y)S \oplus z^nS \to (x-y)S \oplus z^nS; \quad \begin{pmatrix} a(x-y) \\ bz^n \end{pmatrix} \mapsto \begin{pmatrix} bz^n(x-y) \\ byz^n \end{pmatrix},$$

so that the matrix representation of N is $\left(\begin{smallmatrix} 0 & z^n \\ 0 & y \end{smallmatrix}\right)$.

Remark 3.4. Araya, Iima and the second author [1] show that the Cohen-Macaulay modules which appear in Theorem 3.1 (resp. Theorem 3.2) are obtained from the extension by $R/(x)$ and itself (resp. $R/(x)$ and $R/(y)$). In the case, we have the degeneration by Remark 2.2(2). Summing up this fact, we obtain complete description of the degenerations of indecomposable Cohen-Macaulay modules over $k[\![x,y]\!]/(x^2)$ and $k[\![x,y,z]\!]/(xy)$.

4. Construction of degenerations of modules

In this section we provide a construction of a nontrivial degeneration over an arbitrary commutative noetherian algebra over a field (without matrix representations). We begin with establishing a lemma.

Lemma 4.1. *Let R be a commutative ring, and let $\cdots \xrightarrow{\alpha} L \xrightarrow{\beta} L \xrightarrow{\alpha} L \xrightarrow{\beta} \cdots$ be an exact sequence of R-modules. Let $x \in R$, and set $Z = \alpha(L) + xL$ and $W = \beta(L) + xL$.*

(1) *There is an exact sequence*

$$\begin{tikzcd}
\cdots \arrow[rr, "\left(\begin{smallmatrix} \alpha & x \\ 0 & \beta \end{smallmatrix}\right)"] \arrow[dr] & & L \oplus L \arrow[rr, "\left(\begin{smallmatrix} \beta & -x \\ 0 & \alpha \end{smallmatrix}\right)"] \arrow[dr] & & L \oplus L \arrow[rr, "\left(\begin{smallmatrix} \alpha & x \\ 0 & \beta \end{smallmatrix}\right)"] \arrow[dr] & & L \oplus L \arrow[rr, "\left(\begin{smallmatrix} \beta & -x \\ 0 & \alpha \end{smallmatrix}\right)"] \arrow[dr] & & \cdots . \\
& Z \arrow[ur] & & W \arrow[ur] & & Z \arrow[ur] & & W \arrow[ur] &
\end{tikzcd}$$

(2) *Suppose that x is L-regular. Then there are exact sequences:*

$$L \oplus L \xrightarrow{\left(\begin{smallmatrix} \beta & -x \\ 0 & \alpha \end{smallmatrix}\right)} L \oplus L \xrightarrow{(\alpha \ x)} L \to L/Z \to 0,$$

$$L \oplus L \xrightarrow{\left(\begin{smallmatrix} \alpha & x \\ 0 & \beta \end{smallmatrix}\right)} L \oplus L \xrightarrow{(\beta \ -x)} L \to L/W \to 0.$$

Proof. (1) Clearly, $\left(\begin{smallmatrix} \alpha & x \\ 0 & \beta \end{smallmatrix}\right)\left(\begin{smallmatrix} \beta & -x \\ 0 & \alpha \end{smallmatrix}\right) = 0$. Let $s, t \in L$ with $\left(\begin{smallmatrix} \alpha & x \\ 0 & \beta \end{smallmatrix}\right)\left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$. Then $\alpha(s) + xt = 0 = \beta(t)$, and $t \in \ker\beta = \operatorname{Im}\alpha$. We have $t = \alpha(u)$ for some $u \in L$, and $0 = \alpha(s) + xt = \alpha(s + xu)$, which implies $s + xu \in \ker\alpha = \operatorname{Im}\beta$. Writing $s + xu = \beta(v)$ with $v \in L$, we get $\left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right) = \left(\begin{smallmatrix} \beta(v) - xu \\ \alpha(u) \end{smallmatrix}\right) = \left(\begin{smallmatrix} \beta & -x \\ 0 & \alpha \end{smallmatrix}\right)\left(\begin{smallmatrix} v \\ u \end{smallmatrix}\right)$. Thus $\operatorname{Im}\left(\begin{smallmatrix} \beta & -x \\ 0 & \alpha \end{smallmatrix}\right) = \ker\left(\begin{smallmatrix} \alpha & x \\ 0 & \beta \end{smallmatrix}\right)$. A symmetric argument shows $\operatorname{Im}\left(\begin{smallmatrix} \alpha & x \\ 0 & \beta \end{smallmatrix}\right) = \ker\left(\begin{smallmatrix} \beta & -x \\ 0 & \alpha \end{smallmatrix}\right)$.

(2) Note that $(\alpha \ x)$ is a submatrix of $\left(\begin{smallmatrix} \alpha & x \\ 0 & \beta \end{smallmatrix}\right)$; we have $(\alpha \ x)\left(\begin{smallmatrix} \beta & -x \\ 0 & \alpha \end{smallmatrix}\right) = 0$. Let $s, t \in L$ with $(\alpha \ x)\left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$. Then $\alpha(s) + xt = 0$, and $0 = \beta(\alpha(s) + xt) = x\beta(t)$. Since x is L-regular, $\beta(t) = 0$. The above argument shows $\left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right) \in \operatorname{Im}\left(\begin{smallmatrix} \beta & -x \\ 0 & \alpha \end{smallmatrix}\right)$. Thus $\operatorname{Im}\left(\begin{smallmatrix} \beta & -x \\ 0 & \alpha \end{smallmatrix}\right) = \ker(\alpha \ x)$. A symmetric argument shows $\operatorname{Im}\left(\begin{smallmatrix} \alpha & x \\ 0 & \beta \end{smallmatrix}\right) = \ker(\beta \ -x)$. □

Recall that a finitely generated module M over a commutative noetherian ring R is Cohen-Macaulay if $\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \geq \dim R_\mathfrak{p}$ for all prime ideals $\mathfrak{p}$ of R. The main result of this section is the following.

Theorem 4.2. *Let R be a commutative ring, and let $\cdots \xrightarrow{\alpha} L \xrightarrow{\beta} L \xrightarrow{\alpha} L \xrightarrow{\beta} \cdots$ be an exact sequence of R-modules. Let x be an L-regular element such that $\beta(L) \subseteq \alpha(L) + xL =: Z$. Let M be an R-module with $Z \subseteq M \subseteq L$, and set $N = \beta(M) + xZ$.*

(1) *The sequence*

$$0 \to Z \xrightarrow{\left(\begin{smallmatrix} \theta \\ \eta \end{smallmatrix}\right)} M \oplus Z \xrightarrow{(\beta \ -x)} N \to 0$$

is exact, where θ is the inclusion map and $\eta(\alpha(s) + xt) = \beta(t)$ for $s, t \in L$.

(2) *Suppose that R contains a field, Z, M are finitely generated and β is nilpotent. Then M degenerates to N.*

(3) *Assume R is noetherian. If L, M are Cohen-Macaulay, then so are Z, N.*

Proof. (1) As the image of β is contained in Z, so is that of η. Take any element $z \in Z$. Then $z = \alpha(s)+xt$ for some $s, t \in L$, and we have $\beta(z) = \beta(\alpha(s) + xt) = x\beta(t)$. Since x is L-regular and $\beta(t) \in \beta(L) \subseteq Z$, the assignment $z \mapsto \beta(t)$ gives a map $\eta : Z \to Z$, and

$$\beta(z) = x\eta(z) \text{ for all } z \in Z. \tag{4.1}$$

As θ is injective, so is the map $\left(\begin{smallmatrix}\theta\\ \eta\end{smallmatrix}\right)$, whose image is contained in the kernel of the map $\left(\begin{smallmatrix}\beta & -x\end{smallmatrix}\right)$ because $\left(\begin{smallmatrix}\beta & -x\end{smallmatrix}\right)\left(\begin{smallmatrix}\theta\\ \eta\end{smallmatrix}\right)(\alpha(s) + xt) = \left(\begin{smallmatrix}\beta & -x\end{smallmatrix}\right)\left(\begin{smallmatrix}\alpha(s)+xt\\ \beta(t)\end{smallmatrix}\right) = \beta(xt) - x\beta(t) = 0$. The definition of N shows that $\left(\begin{smallmatrix}\beta & -x\end{smallmatrix}\right)$ is surjective.

It remains to show that the kernel of $\left(\begin{smallmatrix}\beta & -x\end{smallmatrix}\right)$ is contained in the image of $\left(\begin{smallmatrix}\theta\\ \eta\end{smallmatrix}\right)$. Let $\left(\begin{smallmatrix}m\\ z\end{smallmatrix}\right) \in M \oplus Z$ such that $\left(\begin{smallmatrix}\beta & -x\end{smallmatrix}\right)\left(\begin{smallmatrix}m\\ z\end{smallmatrix}\right) = 0$. Lemma 4.1 implies that there exist $s, t \in L$ such that $\left(\begin{smallmatrix}m\\ z\end{smallmatrix}\right) = \left(\begin{smallmatrix}\alpha & x\\ 0 & \beta\end{smallmatrix}\right)\left(\begin{smallmatrix}s\\ t\end{smallmatrix}\right) = \left(\begin{smallmatrix}\alpha(s)+xt\\ \beta(t)\end{smallmatrix}\right) = \left(\begin{smallmatrix}\theta\\ \eta\end{smallmatrix}\right)(\alpha(s) + xt)$. Thus we are done.

(2) The module N is finitely generated, and $\beta^n = 0$ for some $n > 0$. It follows by (4.1) that $0 = \beta^n(z) = (x\eta)^n(z) = x^n\eta^n(z)$ for all $z \in Z$. As x^n is L-regular, we get $\eta^n(z) = 0$, that is, η is nilpotent. We see from (1) and Remark 2.2(1) that M degenerates to N.

(3) Set $W = \beta(L) + xL$. Combining (1) and Lemma 4.1, we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Z \arrow[r] \arrow[d, equal] & L \oplus L \arrow[r, "\left(\begin{smallmatrix}\beta & -x\end{smallmatrix}\right)"] & L \arrow[r] \arrow[d, equal] & L/W \arrow[r] & 0 \\
0 \arrow[r] & Z \arrow[r, "\left(\begin{smallmatrix}\theta\\ \eta\end{smallmatrix}\right)"] & M \oplus Z \arrow[r, "\left(\begin{smallmatrix}\beta & -x\end{smallmatrix}\right)"] \arrow[u, "\left(\begin{smallmatrix}i & 0\\ 0 & j\end{smallmatrix}\right)"] & L \arrow[r] & L/N \arrow[r] & 0
\end{tikzcd}$$

with exact rows, where i, j are inclusion maps. The snake lemma gives an exact sequence

$$0 \to L/M \oplus L/Z \to L/N \to L/W \to 0. \tag{4.2}$$

From now on, assuming that R is noetherian, we show that if L, M are Cohen-Macaulay, then so are Z, N. For this, we may assume that R is a local ring of Krull dimension d. The first exact sequence in Lemma 4.1 shows that Z, W are Cohen-Macaulay. The depth lemma implies that the R-modules $L/M, L/Z, L/W$ have depth at least $d - 1$. It follows from (4.2) that L/N also has depth at least $d - 1$, which implies that N is Cohen-Macaulay. □

As a direct corollary of the theorem, we obtain the following result.

Corollary 4.3. *Let R be a commutative noetherian ring containing a field, and let L be a finitely generated R-module. Let α be an endomorphism of L with $\operatorname{Im}\alpha = \ker\alpha$, and let $x \in R$ be an L-regular element. Then the following statements hold.*

(1) *An R-module M with $\alpha(L) + xL \subseteq M \subseteq L$ degenerates to $N := \alpha(M) + x^2L$.*
(2) *If L, M are Cohen-Macaulay, then so is N.*

Proof. Note $\alpha^2 = 0$. Letting $\alpha = \beta$ in Theorem 4.2 shows that M degenerates to $\alpha(M) + x\alpha(L) + x^2L$, and if the former is Cohen-Macaulay, then so is the latter. As $xL \subseteq M$, we have $x\alpha(L) = \alpha(xL) \subseteq \alpha(M)$. Hence $\alpha(M) + x\alpha(L) + x^2L = N$. □

From now on, we give applications of Corollary 4.3.

Corollary 4.4. *Let R be a commutative noetherian ring containing a field, and let $\cdots \xrightarrow{\alpha} L \xrightarrow{\beta} L \xrightarrow{\alpha} L \xrightarrow{\beta} \cdots$ be an exact sequence of R-modules. Let x be an L-regular element, and set $Z = \alpha(L) + xL$ and $W = \beta(L) + xL$. Let M, N be R-submodules of L containing Z, W respectively. Then $M \oplus N$ degenerates to $K := (\alpha(N) + x^2L) \oplus (\beta(M) + x^2L)$. If L, M and N are Cohen-Macaulay, then so is K.*

Proof. The endomorphism $\gamma = \left(\begin{smallmatrix}0 & \alpha\\ \beta & 0\end{smallmatrix}\right) : L^{\oplus 2} \to L^{\oplus 2}$ is such that $\operatorname{Im}\gamma = \ker\gamma$. Putting $X = \{\left(\begin{smallmatrix}m\\ n\end{smallmatrix}\right) \mid m \in M, n \in N\}$, we have $\gamma(L^{\oplus 2}) + xL^{\oplus 2} \subseteq X \subseteq L^{\oplus 2}$. Note that $X \cong M \oplus N$ and $\gamma(X) + x^2L^{\oplus 2} \cong K$. Now the assertion follows from Corollary 4.3. □

Remark 4.5. Let R be a hypersurface local ring. Let X be a Cohen-Macaulay R-module. Then X has a free resolution of the form $\cdots \xrightarrow{\alpha} F \xrightarrow{\beta} F \xrightarrow{\alpha} F \to X \to 0$, and one can apply Corollary 4.4 to $L = F$.

Corollary 4.6. *Let R be a commutative noetherian ring containing a field and L a finitely generated R-module. Let α be an endomorphism of L with $\operatorname{Im}\alpha = \ker\alpha$, and let $x \in R$ be an L-regular element. Then $\alpha(L) + x^jL$ degenerates to $\alpha(L) + x^{2i-j}L$ for all $i \geq j \geq 0$. If L is Cohen-Macaulay, then so are $\alpha(L) + x^jL$ and $\alpha(L) + x^{2i-j}L$.*

Proof. We have $\alpha(L) + x^iL \subseteq \alpha(L) + x^jL \subseteq L$. Corollary 4.3 implies that $\alpha(L) + x^jL$ degenerates to $\alpha(\alpha(L) + x^jL) + x^{2i}L = x^j\alpha(L) + x^{2i}L \cong \alpha(L) + x^{2i-j}L$, where the isomorphism follows from the fact that x^j is L-regular. Fix $h \geq 0$. Lemma 4.1 implies that $C := \alpha(L) + x^hL$ is isomorphic to the cokernel of $\left(\begin{smallmatrix} \alpha & -x^h \\ 0 & \alpha \end{smallmatrix}\right)$, and that the sequence $\cdots \xrightarrow{\left(\begin{smallmatrix} \alpha & -x^h \\ 0 & \alpha \end{smallmatrix}\right)} L^2 \xrightarrow{\left(\begin{smallmatrix} \alpha & x^h \\ 0 & \alpha \end{smallmatrix}\right)} L^2 \xrightarrow{\left(\begin{smallmatrix} \alpha & -x^h \\ 0 & \alpha \end{smallmatrix}\right)} \cdots$ is exact. Using this exact sequence, we easily observe that for each prime ideal $\mathfrak{p}$ of R, an $L_{\mathfrak{p}}$-sequence is a $C_{\mathfrak{p}}$-sequence. Thus, if L is Cohen-Macaulay, then so is C. □

We present a couple of examples. For such an element x as in the second statement it is said that $[x, x]$ is an exact pair of zerodivisors.

Example 4.7. Let R be a commutative noetherian ring containing a field.
(1) Let L be a finitely generated R-module having a submodule C such that $L/C \cong C$. Let $x \in R$ be an L-regular element. Then, for all $i \geq j \geq 0$, $C + x^jL$ degenerates to $C + x^{2i-j}L$.
(2) Let $x \in R$ be such that $(0 : x) = (x)$, and let $y \in R$ be a non-zerodivisor. Then for all integers $i \geq j \geq 0$ (x, y^j) degenerates to (x, y^{2i-j}), where (x, y^j) and (x, y^{2i-j}) are ideals of R.

Proof. The composition $\alpha : L \twoheadrightarrow L/C \cong C \hookrightarrow L$ satisfies $\operatorname{Im}\alpha = \ker\alpha$. The first assertion is shown by Corollary 4.6. The second assertion follows from the first one. □

Remark 4.8. Example 4.7(2) immediately recovers the degeneration obtained in [3, Remark 2.5] and the 'if' part of Theorem 3.1.

5. Remarks on degenerations over hypersurface rings

In this section, we consider extending Theorem 1.1 to higher dimension. Let k be an algebraically closed field of characteristic not two and let $R = S/(f)$ be a hypersurface, where $S = k[\![x_0, x_1, \cdots, x_n]\!]$ is a formal power series ring with maximal ideal $\mathfrak{m}_S = (x_0, x_1, \cdots, x_n)$ and $f \in \mathfrak{m}_S$. We define $R^\sharp = S[\![u]\!]/(f + u^2)$ and $R^{\sharp\sharp} = S[\![u, v]\!]/(f + u^2 + v^2)$. In what follows, for a certain hypersurface R, we investigate the connection among the degenerations of Cohen-Macaulay modules over R, $R^\sharp$ and $R^{\sharp\sharp}$.

Remark 5.1. Let $R = S[\![y]\!]/(y^2 + f)$ with $f \in \mathfrak{m}_S$.
(1) There is a one-to-one correspondence between

- the isomorphism classes of Cohen-Macaulay R-modules, and
- the equivalence classes of square matrices φ with entries in S such that $\varphi^2 = -f$.

This follows by considering matrix representations over S.
(2) Let M be a Cohen-Macaulay R-module, and let ΩM be the first syzygy of M with respect to the minimal R-free resolution of M. As remarked in [7, Lemma 12.2], the matrix representation of ΩM over S is $-\mu$, where μ is that of M.

Let R be as in the above remark. Let μ be a square matrix with entries in S such that $\mu^2 = -f$ and M a Cohen-Macaulay R-module which corresponds to μ. Consider the matrices $\mu^\sharp = \left(\begin{smallmatrix} \mu & u \\ -u & -\mu \end{smallmatrix}\right)$ and $\mu^{\sharp\sharp} = \left(\begin{smallmatrix} \mu & \zeta \\ -\eta & -\mu \end{smallmatrix}\right)$, where $\zeta = u + \sqrt{-1}v$ and $\eta = u - \sqrt{-1}v$. By Remark 5.1(1) we have $(\mu^\sharp)^2 = -(f + u^2)$ (resp. $(\mu^{\sharp\sharp})^2 = -(f + u^2 + v^2)$), and $\mu^\sharp$ (resp. $\mu^{\sharp\sharp}$) corresponds to a Cohen-Macaulay module over $R^\sharp$ (resp. $R^{\sharp\sharp}$), which we denote by $M^\sharp$ (resp. $M^{\sharp\sharp}$). Note that $(M^\sharp)^\sharp$ is not always isomorphic to $M^{\sharp\sharp}$.

To show our main result, we state and prove a lemma.

Lemma 5.2. *Let R be a commutative noetherian ring and M, N finitely generated R-modules. If M degenerates to N, then M/xM also degenerates to N/xN for an N-regular element x.*

Proof. Suppose that M degenerates to N along V. As mentioned in Remark 2.2(1), we have an exact sequence $0 \to Z \to M \oplus Z \to N \to 0$, where the endomorphism g of Z is nilpotent. Since x is N-regular, we also have $0 \to Z/xZ \to M/xM \oplus Z/xZ \to N/xN \to 0$. The endomorphism $g \otimes R/xR$ is also nilpotent, so that M/xM degenerates to N/xN along V. □

The following is the main result of this section.

Proposition 5.3. *Let $R = S[\![y]\!]/(y^2+f)$ be a hypersurface, and let M, N be Cohen-Macaulay R-modules. Consider the following six conditions.*

(1) *M degenerates to N.* (2) *$M^{\sharp\sharp}$ degenerates to $N^{\sharp\sharp}$.* (3) *$M^{\sharp}$ degenerates to $N^{\sharp}$.*
(1') *M degenerates to ΩN.* (2') *$M^{\sharp\sharp}$ degenerates to $(\Omega N)^{\sharp\sharp}$.* (4) *$M \oplus \Omega M$ degenerates to $N \oplus \Omega N$.*

Then the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) *and* (1') $\Rightarrow$ (2') $\Rightarrow$ (3) $\Rightarrow$ (4) *hold true.*

Proof. (1) $\Rightarrow$ (2): Assume that M degenerates to N along V. Let μ and ν be the matrix representations of M and N over S. We have the matrix representation ξ over $S \otimes_k V$ such that $\xi \otimes_V V_t \cong \mu \otimes_k V_t$ and $\xi \otimes_V V/tV \cong \nu$. Now consider the matrix $\xi^{\sharp\sharp} = \begin{pmatrix} \xi & \zeta \\ -\eta & -\xi \end{pmatrix}$, where $\zeta = u + \sqrt{-1}v$ and $\eta = u - \sqrt{-1}v$. Then one can show that $\xi^{\sharp\sharp} \otimes_V V_t \cong \mu^{\sharp\sharp}$ and $\xi^{\sharp\sharp} \otimes_V V/tV \cong \nu^{\sharp\sharp}$. In fact, since $\alpha^{-1}(\xi \otimes_V V_t)\alpha = \mu$ and $\beta^{-1}(\xi \otimes_V V/tV)\beta = \nu$ respectively for some suitable invertible matrices α and β, we have

$$\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha^{-1} \end{pmatrix} (\xi^{\sharp\sharp} \otimes_V V_t) \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} = \begin{pmatrix} \alpha^{-1}(\xi\otimes_V V_t)\alpha & \alpha^{-1}(\zeta\otimes_V V_t)\alpha \\ \alpha^{-1}(-\eta\otimes_V V_t)\alpha & \alpha^{-1}(-\xi\otimes_V V_t)\alpha \end{pmatrix} = \begin{pmatrix} \mu & \zeta \\ -\eta & -\mu \end{pmatrix} = \mu^{\sharp\sharp}.$$

Similarly,

$$\begin{pmatrix} \beta^{-1} & 0 \\ 0 & \beta^{-1} \end{pmatrix} (\xi^{\sharp\sharp} \otimes_V V/tV) \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} \beta^{-1}(\xi\otimes_V V/tV)\beta & \beta^{-1}(\zeta\otimes_V V/tV)\beta \\ \beta^{-1}(-\eta\otimes_V V/tV)\beta & \beta^{-1}(-\xi\otimes_V V/tV)\beta \end{pmatrix} = \begin{pmatrix} \nu & \zeta \\ -\eta & -\nu \end{pmatrix} = \nu^{\sharp\sharp}.$$

According to Corollary 2.7, we conclude that $M^{\sharp\sharp}$ degenerates to $N^{\sharp\sharp}$.

(1′) $\Rightarrow$ (2′): The similar argument to the proof of (1) $\Rightarrow$ (2) shows this.

(2) $\Rightarrow$ (3): By Lemma 5.2, we see that $M^{\sharp\sharp}/vM^{\sharp\sharp}$ degenerates to $N^{\sharp\sharp}/vN^{\sharp\sharp}$. Clearly, $M^{\sharp\sharp}/vM^{\sharp\sharp} \cong M^{\sharp}$ and $N^{\sharp\sharp}/vN^{\sharp\sharp} \cong N^{\sharp}$. Thus we obtain the assertion.

(2′) $\Rightarrow$ (3): We also have the degeneration from $M^{\sharp\sharp}/vM^{\sharp\sharp} \cong M^{\sharp}$ to $(\Omega N)^{\sharp\sharp}/v(\Omega N)^{\sharp\sharp}$ by Lemma 5.2. We note that $(\Omega N)^{\sharp\sharp}/v(\Omega N)^{\sharp\sharp} \cong N^{\sharp}$. Actually, for the matrix representation $\begin{pmatrix} -\nu & u \\ -u & \nu \end{pmatrix}$ of $(\Omega N)^{\sharp\sharp}/v(\Omega N)^{\sharp\sharp}$, we have $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\nu & u \\ -u & \nu \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \nu & u \\ -u & -\nu \end{pmatrix}$. Thus, the implication holds.

(3) $\Rightarrow$ (4): It follows from Lemma 5.2 since $M^{\sharp}/uM^{\sharp} \cong M \oplus \Omega M$ and $N^{\sharp}/uN^{\sharp} \cong N \oplus \Omega N$. □

Remark 5.4. In Proposition 5.3, (4) does not necessarily imply (1) or (1'). Let $R = k[\![x, y, z]\!]/(x^3 + y^2 + z^2)$ and $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$ be ideals (x, y), (x, y^2), (x, y^3) of R respectively. Then we have the exact sequence $0 \to \mathfrak{q} \to \mathfrak{p} \oplus \mathfrak{r} \to \mathfrak{q} \to 0$, so that $\mathfrak{p} \oplus \mathfrak{r}$ degenerates to $\mathfrak{q} \oplus \mathfrak{q}$. Notice that $\mathfrak{r} \cong \Omega\mathfrak{p}$ and $\mathfrak{q} \cong \Omega\mathfrak{q}$. However $\mathfrak{q}$ never degenerates to $\mathfrak{p}$ or $\mathfrak{q}$. See [3, Theorem 3.1] for the details.

Recall that Knörrer's periodicity theorem [9, Theorem 12.10] gives an equivalence $\Phi : \underline{\mathrm{CM}}(R) \to \underline{\mathrm{CM}}(R^{\sharp\sharp})$ of triangulated categories. We call this functor a *Knörrer's periodicity functor*. Using Proposition 5.3, we generalize the 'if' part of Theorem 3.1.

Example 5.5. Consider the hypersurface singularity of type (A_∞) with odd dimension:

$$k[\![x_0, x_1, \dots, x_n]\!]/(x_1^2 + x_2^2 + \cdots + x_n^2).$$

Let $M(h)$ be the image of (x_0, z^h) by the composition of Knörrer's periodicity functors $\underline{\mathrm{CM}}(k[\![x_0, x_1]\!]/(x_1^2)) \to \underline{\mathrm{CM}}(k[\![x_0, x_1, x_2, x_3]\!]/(x_1^2 + x_2^2 + x_3^2)) \to \cdots \to \underline{\mathrm{CM}}(k[\![x_0, x_1, \dots, x_n]\!]/(x_1^2 + x_2^2 + \cdots + x_n^2))$. Iterated application of Proposition 5.3 shows that $M(j)$ degenerates to $M(2i - j)$.

Proposition 5.6. *Let $R = S/(f)$ with $f \in \mathfrak{m}_S$ and let $\alpha : S^n \to S^n$ be an endomorphism such that (α, α) is a matrix factorization of f over S. Let $z \in R$ be a non-zerodivisor, and consider the map $\begin{pmatrix} \alpha & z^h \end{pmatrix} : R^n \oplus R^n \to R^n$ for each $h \geq 0$. Set $\zeta = u + \sqrt{-1}v$ and $\eta = u - \sqrt{-1}v$. Then the following hold.*

(a) *The images $\mathrm{Im}\begin{pmatrix} \alpha & z^h \end{pmatrix}$ and $\mathrm{Im}\begin{pmatrix} \alpha & \zeta & z^h & 0 \\ \eta & -\alpha & 0 & z^h \end{pmatrix}$ are Cohen-Macaulay modules over R and $R^{\sharp\sharp}$, respectively.*

(b) *The functor Φ sends $\mathrm{Im}\begin{pmatrix} \alpha & z^h \end{pmatrix}$ to $\mathrm{Im}\begin{pmatrix} \alpha & \zeta & z^h & 0 \\ \eta & -\alpha & 0 & z^h \end{pmatrix}$.*

(c) *The R-module $\mathrm{Im}\begin{pmatrix} \alpha & z^h \end{pmatrix}$ is nonfree and indecomposable if and only if so is the $R^{\sharp\sharp}$-module $\mathrm{Im}\begin{pmatrix} \alpha & \zeta & z^h & 0 \\ \eta & -\alpha & 0 & z^h \end{pmatrix}$.*

(d) *For all $i \geq j \geq 0$ one can show that $\mathrm{Im}\begin{pmatrix} \alpha & z^j \end{pmatrix}$ degenerates to $\mathrm{Im}\begin{pmatrix} \alpha & z^{2i-j} \end{pmatrix}$ and $\mathrm{Im}\begin{pmatrix} \alpha & \zeta & z^j & 0 \\ \eta & -\alpha & 0 & z^j \end{pmatrix}$ degenerates to $\mathrm{Im}\begin{pmatrix} \alpha & \zeta & z^{2i-j} & 0 \\ \eta & -\alpha & 0 & z^{2i-j} \end{pmatrix}$ as Cohen-Macaulay modules over R, $R^{\sharp\sharp}$ respectively.*

Proof. We regard α and $\alpha' = \begin{pmatrix} \alpha & \zeta \\ \eta & -\alpha \end{pmatrix}$ as the endomorphisms of R^n and $R^{\sharp\sharp 2n}$, respectively. Note that $\operatorname{Im}\alpha = \ker\alpha$ and $\operatorname{Im}\alpha' = \ker\alpha'$. Assertions (a) and (d) follow from Corollary 4.6. By [7, (12.8.1)], the cokernel of $\begin{pmatrix} \alpha & -z^h \\ 0 & \alpha \end{pmatrix}$ is sent by Φ to that of $\begin{pmatrix} \alpha & -z^h & \zeta & 0 \\ 0 & \alpha & 0 & \zeta \\ \eta & 0 & -\alpha & -z^h \\ 0 & \eta & 0 & -\alpha \end{pmatrix}$, which is transformed into $\begin{pmatrix} \alpha & \zeta & -z^h & 0 \\ \eta & -\alpha & 0 & -z^h \\ 0 & 0 & \alpha & \zeta \\ 0 & 0 & \eta & -\alpha \end{pmatrix}$ by elementary row and column operations. By Lemma 4.1, the cokernel of the latter matrix is the image of $\begin{pmatrix} \alpha & \zeta & z^h & 0 \\ \eta & -\alpha & 0 & z^h \end{pmatrix}$. Thus (b) follows. As Φ is an equivalence of additive categories, (c) follows from (b). $\square$

Remark 5.7. (1) Since, for each $h > 0$, the ideal (x_0, z^h) of the ring $k[\![x_0, z]\!]/(x_0^2)$ is nonfree and indecomposable and coincides with $\operatorname{Im}\begin{pmatrix} x_0 & z^h \end{pmatrix}$, Example 5.5 can also be obtained from the above proposition. (2) Let R be an algebra over a field. Let $0 \to L \to M \to N \to 0$ be an exact sequence of finitely generated R-modules. Then M degenerates to $L \oplus N$ by Remark 2.2(2). Such a degeneration is called a *degeneration by an extension* in [3, Definition 2.4]. Thus, it is important to investigate the existence of degenerations that cannot be obtained by extensions. The degeneration in Example 5.5 cannot be obtained by (iterated) extensions, because $M(2i-j)$ is an indecomposable R-module.

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