HOMOTOPY CATEGORIES OF UNBOUNDED COMPLEXES OF PROJECTIVE MODULES

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Abstract. We develop in this paper the stable theory for projective complexes, by which we mean to consider a chain complex of finitely generated projective modules as an object of the factor category of the homotopy category modulo split complexes. As a result of the stable theory we are able to prove that any complex of finitely generated projective modules over a generically Gorenstein ring is acyclic if and only if its dual complex is acyclic. This shows the dependence of total reflexivity conditions for modules over a generically Gorenstein ring.

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1. Introduction

In this paper we are mainly interested in unbounded cochain complexes consisting of finitely generated projective modules over a commutative Noetherian ring. Of most interest to us in the present paper are the properties of complexes that are independent of any additional split summands. For this purpose we develop the stable theory for those complexes. For the module category such an idea was first proposed and

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established by Auslander-Bridger [2] under the name of 'stable module theory'. We apply their idea to the homotopy category of complexes of finitely generated projective modules.

The whole of our stable theory for complexes is devoted to prove the following single theorem.

Theorem 1.1 (See Theorem 11.7)**.** *Let R be a commutative Noetherian ring that is assumed to be a generically Gorenstein ring, and X a complex of finitely generated projective R-modules. Then, X is acyclic (i.e. $H(X) = 0$) if and only if the R-dual X^* is acyclic (i.e. $H(X^*) = 0$).*

Recall that a commutative Noetherian ring is called a generically Gorenstein ring if the total ring of quotients is a Gorenstein ring, or equivalently $R_{\mathfrak{p}}$ is a Gorenstein ring for every associated prime $\mathfrak{p} \in \mathrm{Ass}(R)$. As a matter of fact, every Noetherian integral domain, a little more generally, every reduced Noetherian ring, is a generically Gorenstein ring. A similar theorem to Theorem 1.1, but under a more special setting, was considered in [14, Corollary 1.4].

Analogously to the stable module theory of Auslander and Bridger, we will introduce the parallel notion of torsion-freeness and reflexivity for complexes, which we call *torsion-free complexes and *reflexive complexes in this paper (Definition 3.1). We observe in Theorem 2.3 that there is an exact sequence similar to the Auslander-Bridger sequence. If the base ring R is a generically Gorenstein ring, then as we shall show in Theorem 4.2, a complex X is *torsion-free if and only if the cohomology modules $H^i(X^*)$ ($i \in \mathbb{Z}$) are torsion-free as R-modules.

A crucial point for the proof of Theorem 1.1 is how one can relate a generic condition of the ring such as the generic Gorenstein condition with the *torsion-free or the *reflexive property for complexes. This will be accomplished by considering the factor category of the homotopy category. To be more precise, let $\mathscr{K}(R)$ be the homotopy category of all complexes of finitely generated projective modules over a commutative Noetherian ring R, and let $\mathrm{Add}(R)$ be its additive full subcategory consisting of all split complexes. See Definition 5.2 and Theorem 5.8 for further details. We show in Lemmas 7.2 and 7.5 that $\mathrm{Add}(R)$ is functorially finite in $\mathscr{K}(R)$ and hence every complex in $\mathscr{K}(R)$ can be resolved by complexes in $\mathrm{Add}(R)$.

We define $\underline{\mathscr{K}(R)}$ to be the factor category $\mathscr{K}(R)/\mathrm{Add}(R)$ and call it the stable category. Then we are able to define the syzygy functor Ω and the cosyzygy functor Σ on $\underline{\mathscr{K}(R)}$, and as a result we have an adjoint pair (Σ, Ω) of functors (Theorem 7.11). Thus there is a natural counit morphism $\pi_X^n : \Sigma^n\Omega^n X \to X$ for any positive integer n and for any complex X. In terms of syzygy functors, we can characterize the *torsion-free property for X as the counit morphism π_X^1 is an isomorphism in $\underline{\mathscr{K}(R)}$ (Theorem 7.14). We develop in §8 some new idea to construct complexes by successive use of mapping cone constructions, which we shall call the contraction. See Theorem and Definition 8.2.

Now taking the mapping cones of the counit morphisms in $\mathscr{K}(R)$, we have triangles of the form

$$\Delta^{(n,0)}(X) \longrightarrow \Sigma^n\Omega^n(X) \xrightarrow{\pi_X^n} X \longrightarrow \Delta^{(n,0)}(X)[1],$$

by which we define the complexes $\Delta^{(n,0)}(X)$ for X. Then we shall show that all such complexes $\Delta^{(n,0)}(X)$ have a finite $\mathrm{Add}(R)$-resolution of length at most $n-1$. See Theorem 10.2 for more precise statement, which is one of the key theorem in order to prove Theorem 1.1. After observing these facts, we prove in Theorem 11.1 that any syzygy complex $\Omega^r X$ is *torsion-free if $H(X^*) = 0$. This is the second key theorem to prove Theorem 1.1. As a consequence of this theorem, we are eventually able to prove the main theorem 1.1 in §11.

The following are the corollaries that are proved straightforwardly from Theorem 1.1, and each one is proved in the last section of this paper.

Corollary 1.2. *Assume that the ring R is a generically Gorenstein ring. Let $f : X \to Y$ be a chain homomorphism between complexes of finitely generated projective modules over R. Then, f is a quasi-isomorphism if and only if the R-dual $f^* : Y^* \to X^*$ is a quasi-isomorphism.*

Corollary 1.3. *Assume that the ring R is a generically Gorenstein ring. Let M be a finitely generated R-module. Then the following conditions are equivalent:*

(1) *M is a totally reflexive R-module.*

(2) $\mathrm{Ext}^i_R(M,R) = 0$ *for all $i > 0$.*

(3) *M is an infinite syzygy, i.e. there is an exact sequence of infinite length of the form $0 \longrightarrow M \longrightarrow P_0 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow \cdots$, where each P_i is a finitely generated projective R-module.*

Corollary 1.4. *Under the assumption that R is a generically Gorenstein ring, we have the equality of G-dimension;*

$$\text{G-dim}_R M = \sup\{n \in \mathbb{Z} \mid \mathrm{Ext}^n_R(M,R) \neq 0\ \},$$

for a finitely generated R-module M.

Jorgensen and Şega [8] gave an example of a module over a non-Gorenstein Artinian ring that disproves the implication (2) ⇒ (1) in Corollary 1.3, hence the generic Gorensteinness assumption in the theorem is indispensable.

The following is a commutative version of Tachikawa conjecture that is also a consequence of Theorem 1.1. It should be noted that this has been proved by Avramov, Buchweitz and Şega [3].

Corollary 1.5. *Let R be a Cohen-Macaulay ring with canonical module ω. Furthermore assume that R is a generically Gorenstein ring. If $\mathrm{Ext}^i_R(\omega,R) = 0$ for all $i > 0$, then R is Gorenstein.*

Corollary 1.6. *Assume that the ring R is a generically Gorenstein ring. Let X be a complex of finitely generated projective modules. Assume both $H(X)$ and $H(X^*)$ are bounded above (i.e. $X, X^* \in D^-(R)$). Then we have the isomorphism in the derived category:*

$$X \cong \mathrm{RHom}_R(\mathrm{RHom}_R(X, R), R).$$

Corollary 1.7. *Assume that the ring R is a generically Gorenstein ring. Let X be a complex of finitely generated projective modules. If all the cohomology modules $H^i(X)$ $(i \in \mathbb{Z})$ have dimension at most ℓ as R-modules, then so are the modules $H^i(X^*)$ $(i \in \mathbb{Z})$. In particular, X has cohomology modules of finite length if and only if so does X^*.*

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2. Preliminary observation for complexes

Throughout this paper we assume that R is a commutative Noetherian ring. We denote by $\mathrm{mod}(R)$ the abelian category of finitely generated R-modules and R-module homomorphisms. Furthermore we denote by $\mathrm{proj}(R)$ the additive subcategory of $\mathrm{mod}(R)$, which consists of all finitely generated projective R-modules.

We denote by $\mathscr{C}(R) = C(\mathrm{proj}(R))$ the additive category of complexes over $\mathrm{proj}(R)$ and chain homomorphisms. We also denote by $\mathscr{K}(R) = K(\mathrm{proj}(R))$ the homotopy category consisting of complexes over $\mathrm{proj}(R)$. Note by recalling the definition that objects of $\mathscr{C}(R)$ and $\mathscr{K}(R)$ are complexes consisting of finitely generated projective modules, which we denote cohomologically such as

$$X = \left[\cdots \longrightarrow X^{i-1} \xrightarrow{d_X^{i-1}} X^i \xrightarrow{d_X^i} X^{i+1} \longrightarrow \cdots \right],$$

where each X^i belongs to $\mathrm{proj}(R)$. All cohomology modules $H^i(X)$ $(i \in \mathbb{Z})$ are necessarily finitely generated R-modules for $X \in \mathscr{C}(R)$. A morphism $X \to Y$ in $\mathscr{C}(R)$ is a cochain homomorphism, while a morphism $X \to Y$ in $\mathscr{K}(R)$ is a homotopy equivalence class of a chain homomorphism from X to Y, i.e.

$$\mathrm{Hom}_{\mathscr{K}(R)}(X, Y) = \mathrm{Hom}_{\mathscr{C}(R)}(X, Y)/\{\text{chain homotopy}\}.$$

Both of $\mathrm{Hom}_{\mathscr{C}(R)}(X, Y)$ and $\mathrm{Hom}_{\mathscr{K}(R)}(X, Y)$ have natural R-module structures. However they are not necessarily finitely generated R-modules in general.

Eg. Consider the endomorphisms of the complex $\left[\cdots \longrightarrow R \xrightarrow{0} R \xrightarrow{0} R \longrightarrow \cdots \right]$.

Note that $\mathscr{C}(R)$ is an Abelian category. Note also that a complex $X \in \mathscr{C}(R)$ is the zero object as an object of $\mathscr{K}(R)$ if and only if it is a split exact sequence as a long exact sequence, which is called a null complex. (It is also known as a contractible complex.) Every complex $X \in \mathscr{C}(R)$ has a direct sum decomposition in $\mathscr{C}(R)$ such as $X = X' \oplus N$, where N is a null complex and X' contains no null complex as a direct summand. We should note that such a decomposition is not unique in general.

It is clear and well-known that a chain homomorphism f in $\mathscr{C}(R)$ factors through a null complex if and only if f is null homotopic. Therefore the category $\mathscr{K}(R)$ is a residue category of $\mathscr{C}(R)$ by the ideal generated by the object set consisting of all null complexes. It is easy to verify that $\mathscr{C}(R)$ is a Frobenius category with null complexes as relatively projective and injective objects. In such a sense $\mathscr{K}(R)$ has a structure of triangulated category. Recall that the shift functor $X \mapsto X[1]$ is defined as $X[1]^n = X^{n+1}$ ans $d^n_{X[1]} = -d^{n+1}_X$. Furthermore there is a triangle $X \to Y \to Z \to X[1]$ in $\mathscr{K}(R)$ if and only if there is an exact sequence in $\mathscr{C}(R)$ of the form;

$$0 \longrightarrow X \longrightarrow Y \oplus N \longrightarrow Z \longrightarrow 0,$$

where N is a null complex. One can find such description of triangles in Happel [6, Chapter 1]. The more general references for complexes and triangulated categories are Weibel [12] and Mannin-Gelfand [10].

A remarkable advantage of $\mathscr{K}(R)$ is that it possesses a duality. For $X \in \mathscr{K}(R)$ we are able to define the dual complex by

$$X^* = \mathrm{Hom}_R(X, R), \quad d^n_{X^*} = \mathrm{Hom}_R(d^{-n}_X, R).$$

Note that X^* is again an object of $\mathscr{K}(R)$, since the dual of a finitely generated projective module is finitely generated projective. It is easy to see that the dual functor

$$(-)^* : \mathscr{K}(R) \longrightarrow \mathscr{K}(R)^{\mathrm{op}}, \quad X \mapsto X^*$$

is a triangle functor between triangulated categories. Since X^{**} is naturally isomorphic to X, it actually yields the duality on $\mathscr{K}(R)$.

Notation 2.1. For a complex $X \in \mathscr{K}(R)$, we denote by $C(X)$ the cokernel of the differential mapping $d_X : X \to X[1]$, which is a graded R-module as it is decomposed as $C(X) = \bigoplus_{i\in\mathbb{Z}} C^i(X)$. Similarly the cocycle $Z(X) = \bigoplus_{i\in\mathbb{Z}} Z^i(X)$ is the kernel of d_X and the coboundary $B(X) = \bigoplus_{i\in\mathbb{Z}} B^i(X)$ is the image of d_X.

As $C(X) = X/B(X)$ there is a short exact sequence of graded R-modules such as

$$0 \longrightarrow H(X) \longrightarrow C(X) \longrightarrow B(X)[1] \longrightarrow 0. \tag{2.1}$$

Let X be a complex in $\mathscr{K}(R)$ and let M be an R-module. We denote by $K(\mathrm{Mod}(R))$ the homotopy category of all complexes of any R-modules, and we regard M as a complex concentrated in degree zero. Recall that $\mathrm{Hom}_R(X, M)$ is the Hom complex and an element of the cohomology modules of this complex is nothing but the homotopy class of a chain map from X to M, i.e.

$$H^{-i}(\mathrm{Hom}_R(X, M)) = \mathrm{Hom}_{K(\mathrm{Mod}(R))}(X[i], M).$$

Definition 2.2. Let $X \in \mathscr{K}(R)$, M an R-module and $i \in \mathbb{Z}$. As noted above, each element $[f] \in H^{-i}(\mathrm{Hom}_R(X, M))$ is a homotopy class of a chain map $f : X[i] \to M$, thus it induces a unique R-module homomorphism $H^0(f) : H^0(X[i]) = H^i(X) \to H^0(M) = M$, hence an element $H^0(f) \in \mathrm{Hom}_R(H^i(X), M)$. We define an R-module homomorphism

$$\rho^i_{X,M} : H^{-i}(\mathrm{Hom}_R(X, M)) \longrightarrow \mathrm{Hom}_R(H^i(X), M)$$

by $\rho^i_{X,M}([f]) = H^0(f)$.

Theorem 2.3. *Under the circumstances in Definition 2.2, there is an exact sequence of R-modules;*

$$0 \to \mathrm{Ext}^1_R(C^{i+1}(X), M) \to H^{-i}(\mathrm{Hom}_R(X, M)) \xrightarrow{\rho^i_{X,M}} \mathrm{Hom}_R(H^i(X), M) \to \mathrm{Ext}^2_R(C^{i+1}(X), M),$$

for each $i \in \mathbb{Z}$.

Proof. We see from the exact sequence (2.1) that there exists an exact sequence;

(2.2)

$$0 \to \mathrm{Hom}_R(B^{i+1}(X), M) \to \mathrm{Hom}_R(C^i(X), M) \to \mathrm{Hom}_R(H^i(X), M) \to \mathrm{Ext}^1_R(B^{i+1}(X), M),$$

where we should note that $\mathrm{Ext}^1_R(B^{i+1}(X), M) \cong \mathrm{Ext}^2_R(C^{i+1}(X), M)$, since $X^{i+1}/B^{i+1}(X) \cong C^{i+1}(X)$ and X^{i+1} is projective.

Note that $\mathrm{Hom}_R(X, M)^{-i} = \mathrm{Hom}_R(X^i, M)$ for all $i \in \mathbb{Z}$. Thus, from the exact sequence $X^{i-1} \to X^i \to C^i(X) \to 0$, it follows that $0 \to \mathrm{Hom}_R(C^i(X), M) \to \mathrm{Hom}_R(X, M)^{-i} \to \mathrm{Hom}_R(X, M)^{-i+1}$ is exact, hence we have an isomorphism

$$\lambda : Z^{-i}(\mathrm{Hom}_R(X, M)) \to \mathrm{Hom}_R(C^i(X), M),$$

where the left hand side is the $(-i)$th cocycle module of the complex $\mathrm{Hom}_R(X, M)$. On the other hand, the $(-i)$th coboundary $B^{-i}(\mathrm{Hom}_R(X, M))$ is the image the mapping $\mathrm{Hom}_R(X^{i+1}, M) \to \mathrm{Hom}_R(X^i, M)$. From the exact sequence $0 \to B^{i+1}(X) \to X^{i+1} \to C^{i+1}(X) \to 0$, we have an exact sequence

$$\mathrm{Hom}_R(X^{i+1}, M) \xrightarrow{\nu} \mathrm{Hom}_R(B^{i+1}(X), M) \longrightarrow \mathrm{Ext}^1_R(C^{i+1}(X), M) \longrightarrow 0.$$

Since there is a commutative diagram

$$\begin{array}{ccc} X^i & \xrightarrow{d^i_X} & X^{i+1} \\ & \searrow & \uparrow \\ & & B^{i+1}(X), \end{array}$$

we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_R(X^{i+1}, M) & \xrightarrow{(d^i_X)^*} & \mathrm{Hom}_R(X^i, M) \\ \downarrow \nu & \nearrow & \\ \mathrm{Hom}_R(B^{i+1}(X), M), & & \end{array}$$

from which we see that the mapping ν has the image $B^{-i}(\mathrm{Hom}_R(X,M))$. Hence we have an exact sequence

$$0 \longrightarrow B^{-i}(\mathrm{Hom}_R(X,M)) \longrightarrow \mathrm{Hom}_R(B^{i+1}(X),M) \longrightarrow \mathrm{Ext}^1_R(C^{i+1}(X),M) \longrightarrow 0.$$

Combine all the mappings together, we finally have a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
0 \to & B^{-i}(\mathrm{Hom}_R(X,M)) & \to & Z^{-i}(\mathrm{Hom}_R(X,M)) & \to & H^{-i}(\mathrm{Hom}_R(X,M)) & \longrightarrow & 0 \\
 & \downarrow & & \downarrow \lambda & & \downarrow \rho & & \\
0 \to & \mathrm{Hom}_R(B^{i+1}(X),M) & \to & \mathrm{Hom}_R(C^{i}(X),M) & \to & \mathrm{Hom}_R(H^{i}(X),M) & \to & \mathrm{Ext}^2_R(C^{i+1}(X),M) \\
 & \downarrow & & & & & & \\
 & \mathrm{Ext}^1_R(C^{i+1}(X),M) & & & & & & \\
 & \downarrow & & & & & & \\
 & 0 & & & & & &
\end{array}$$

Since λ is an isomorphism, we have the desired exact sequence by the snake lemma. $\square$

Definition 2.4. Let $X \in \mathscr{K}(R)$, $M \in \mathrm{mod}(R)$ and $i \in \mathbb{Z}$. Similarly to Definition 2.2, $X \otimes_R M$ is the tensor complex and we can define a natural R-module homomorphism

$$\sigma^i_{X,M} : H^i(X) \otimes_R M \longrightarrow H^i(X \otimes_R M).$$

Now the similar argument to the previous theorem proves the following theorem.

Theorem 2.5. *There is an exact sequence of R-modules;*

$$\mathrm{Tor}^R_2(C^{i+1}(X),M) \to H^i(X) \otimes_R M \xrightarrow{\sigma^i_{X,M}} H^i(X \otimes_R M) \to \mathrm{Tor}^R_1(C^{i+1}(X),M) \to 0.$$

Example 2.6. Let M be a finitely generated R-module and let

$$P_1 \xrightarrow{f} P_0 \longrightarrow M \longrightarrow 0$$

be a projective presentation of M, where each P_i are finitely generated and projective. Recall that the transpose $\mathrm{Tr}(M)$ of M is defined to be the cokernel of the dual mapping $f^* = \mathrm{Hom}_R(f,R)$ of f.

Now, set the complex X to be $\left[\, 0 \longrightarrow P_0^* \xrightarrow{f^*} P_1^* \longrightarrow 0 \,\right]$. Then we have that $C^1(X) = \mathrm{Tr}(M)$ and $X^* = \left[\, 0 \longrightarrow P_1 \xrightarrow{f} P_0 \longrightarrow 0 \,\right]$. Therefore $H^0(X^*) = M$ and $H^0(X)^* = M^{**}$ in this case. It is easily verified that the mapping ρ^0_{XR} is the natural mapping $M \to M^{**}$. Thus applying Theorem 2.3, we have an exact sequence

$$0 \longrightarrow \mathrm{Ext}^1(\mathrm{Tr}(M),R) \longrightarrow M \longrightarrow M^{**} \longrightarrow \mathrm{Ext}^2(\mathrm{Tr}(M),R),$$

as shown in [2, Chapter 2].

3. *Torsion-free and *reflexive complexes

Definition 3.1. Let $X \in \mathscr{K}(R)$. We denote by X^* the R-dual complex $\mathrm{Hom}_R(X, R)$. As we remarked in Definition 2.2, we have a natural mapping

$$\rho^i_{X,R} : H^{-i}(X^*) \to H^i(X)^*$$

for all $i \in \mathbb{Z}$. We say that the complex X is ***torsion-free** if $\rho^i_{X,R}$ are injective mappings for all $i \in \mathbb{Z}$. Likewise, we say that X is ***reflexive** if $\rho^i_{X,R}$ are isomorphisms for all $i \in \mathbb{Z}$.

Lemma 3.2. *Let X be a complex in $\mathscr{K}(R)$.*

(1) *X is *torsion-free if and only if it satisfies the following condition:*

  ($*$) *For any chain map $f : X \to R[i]$ with $i \in \mathbb{Z}$, if $H(f) = 0$ then $f = 0$ as a morphism in $\mathscr{K}(R)$.*

(2) *Assume that X satisfies the condition ($*$). Then X is *reflexive if and only if it satisfies the following condition:*

  ($**$) *If $a : H^{-i}(X) \to R$ is an R-module homomorphism where $i \in \mathbb{Z}$, then there is a chain map $f : X \to R[i]$ such that $H^{-i}(f) = a$.*

Proof. This is just a restatement of the definition. □

Remark 3.3.

(1) Let X be *torsion-free (resp. *reflexive). Then so are any shifted complexes $X[i]$ for $i \in \mathbb{Z}$. Any direct summands of X are also *torsion-free (resp. *reflexive).
(2) Any direct sums of *torsion-free complexes are *torsion-free. (As we will see in Section 5, the category $\mathscr{K}(R)$ admits certain kind of infinite direct sums. This remark says that if $\{X_i \mid i \in I\}$ is a set of *torsion-free complexes and if $X = \coprod_{i \in I} X_i$ exists in $\mathscr{K}(R)$, then X is also *torsion-free. The proof is clear from Lemma 3.2(1))
(3) Any direct sums of finite number of *reflexive complexes are *reflexive.

The following is straightforward from Theorem 2.3.

Theorem 3.4. *Let $X \in \mathscr{K}(R)$.*

(1) *X is *torsion-free if and only if $\mathrm{Ext}^1_R(C(X), R) = 0$.*
(2) *If $\mathrm{Ext}^1_R(C(X), R) = \mathrm{Ext}^2_R(C(X), R) = 0$, then X is *reflexive.*

Note that the converse of (2) is not necessarily true, i.e. that X is *reflexive does not mean $\mathrm{Ext}^2_R(C(X), R) = 0$.

Corollary 3.5. *If R is a Gorenstein ring of dimension zero, then every complex $X \in \mathscr{K}(R)$ is *reflexive (and hence *torsion-free).*

Proof. In this case $\mathrm{Ext}^i_R(-, R) = 0$ for all $i > 0$. □

Example 3.6. Let M be a finitely generated R-module and let

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

be a projective resolution of M with $P_i \in \mathrm{proj}(R)$ for all $i > 0$.

(1) Setting

$$X = \Big[\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0 \Big] \in \mathscr{K}(R),$$

we can easily see that the following three conditions are equivalent:

(i) $\mathrm{Ext}^i_R(M,R) = 0$ for all $i > 0$.
(ii) X is *torsion-free.
(iii) X is *reflexive.

(2) Let $n > 0$ be an integer. Considering the truncation of X, we set

$$X_{(n)} = \Big[0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0 \Big] \in \mathscr{K}(R).$$

Then $X_{(n)}$ is *torsion-free if and only if $\mathrm{Ext}^i_R(M,R) = 0$ for $1 \le i \le n$, while $X_{(n)}$ is *reflexive if and only if $\mathrm{Ext}^i_R(M,R) = 0$ for $1 \le i \le n+1$.

Proposition 3.7. *Let*

$$X \xrightarrow{a} Y \xrightarrow{b} Z \xrightarrow{c} X[1]$$

be a triangle in $\mathscr{K}(R)$.

(1) *Suppose that $H(b)^* : H(Z)^* \to H(Y)^*$ is injective. If X and Z are *torsion-free, then so is Y.*
(2) *Suppose that the sequence $H(Z)^* \xrightarrow{H(b)^*} H(Y)^* \xrightarrow{H(a)^*} H(X)^*$ is exact. If Z is *reflexive and if Y is *torsion-free, then X is *torsion-free.*
(3) *Suppose that the sequence $H(Z)^* \xrightarrow{H(b)^*} H(Y)^* \xrightarrow{H(a)^*} H(X)^*$ is exact. And assume that X and Z are *reflexive and that Y is *torsion-free. Then Y is *reflexive.*

Proof. (1) Let $f : Y \to R[i]$ be a chain map with $i \in \mathbb{Z}$. Assume $H(f) = 0$. Then $H(fa) = H(f)H(a) = 0$. Since X is *torsion-free, it follows that $fa = 0$ in $\mathscr{K}(R)$. Then there is a morphism $g : Z \to R[i]$ such that $f = gb$. Thus we have $0 = H(f) = H(g)H(b) = H(b)^*(H(g))$ and since $H(b)^*$ is injective, it follows $H(g) = 0$. However, since Z is *torsion-free, we have $g = 0$. Therefore $f = gb = 0$.

(2) Let $f : X \to R[i]$ be a chain map for $i \in \mathbb{Z}$ and we assume that $H(f) = 0$. Then, $H(f \cdot c[-1]) = 0$, and it follows that $f \cdot c[-1] = 0$, since Z is *torsion-free. Hence there is a morphism $g : Y \to R[i]$ with $f = ga$. Note that there is a commutative diagram

of graded R-modules with an exact row:

$$\begin{array}{ccccc} H(X) & \xrightarrow{H(a)} & H(Y) & \xrightarrow{H(b)} & H(Z) \\ & & \downarrow{\scriptstyle H(g)} & & \\ & & H(R[i]) = R[i] & & \end{array}$$

Since $H(a)^*(H(g)) = H(g)H(a) = H(f) = 0$, it follows from the assumption that $H(g)$ induces a graded R-module homomorphism $\epsilon : H(Z) \to H(R[i])$ with $H(g) = \epsilon H(b)$. Since Z is *reflexive, there is a chain map $h : Z \to R[i]$ such that $H(h) = \epsilon$. Then, we have $H(g - hb) = H(g) - H(h)H(b) = 0$. Since Y is *torsion-free, it follows that $g = hb$. Thus $f = ga = hba = 0$ as desired.

(3) Let $\alpha : H(Y) \to R$ be any element of $H(Y)^*$. Since $\alpha H(a) \in H(X)^*$ and since X is *reflexive, there is a morphism $f : X \to R$ such that $\alpha H(a) = H(f)$. Then we have $H(f \cdot c[-1]) = \alpha H(a)H(c[-1]) = 0$. Thus it follows from the *torsion-free property of Z that $f \cdot c[-1] = 0$. Then there is a morphism $g : Y \to R$ with $f = ga$. Therefore we have $\alpha H(a) = H(ga) = H(g)H(a)$, or equivalently $(\alpha - H(g))H(a) = 0$. By the exact sequence $H(Z)^* \xrightarrow{H(b)^*} H(Y)^* \xrightarrow{H(a)^*} H(X)^*$, we find an element $\beta \in H(Z)^*$ satisfying $\alpha - H(g) = \beta H(b)$. Since Z is *reflexive, we have $\beta = H(h)$ for some morphism $h : Z \to R$. Thus we have $\alpha = H(g) + \beta H(b) = H(g + hb)$. □

4. Complexes over a generically Gorenstein ring

Note that a finitely generated R-module M is called **torsionless** if it satisfies one of the following equivalent conditions: (See also Example 2.6.)

(1) M is a submodule of a free R-module.

(2) The natural mapping $M \to M^{**}$ is injective.

(3) $\mathrm{Ext}^1_R(\mathrm{Tr}M, R) = 0$.

On the other hand an R-module M is said to be **torsion-free** if the natural mapping $M \to S^{-1}M$ is injective, where S is the multiplicatively closed subset $R \setminus \bigcup_{\mathfrak{p} \in \mathrm{Ass}(R)} \mathfrak{p}$ consisting of all non-zero divisors of R. Note that every torsionless module is torsion-free.

Recall that a Noetherian commutative ring R is said to be **generically Gorenstein** if every localization $R_{\mathfrak{p}}$ for $\mathfrak{p} \in \mathrm{Ass}(R)$ is a Gorenstein local ring, or equivalently the total quotient ring of R is a Gorenstein ring of dimension zero. The following lemma is well-known.

Lemma 4.1. *Let R be a generically Gorenstein ring. Then a finitely generated R-module M is torsionless if and only if M is torsion-free.*

Proof. We have only to prove the 'if 'part of the lemma. Let $S = R \setminus \bigcup_{\mathfrak{p} \in \mathrm{Ass}(R)} \mathfrak{p}$. There is a commutative diagram;

$$\begin{array}{ccc} M & \xrightarrow{\quad\alpha\quad} & \mathrm{Hom}_R(\mathrm{Hom}_R(M,R),R) \\ {\scriptstyle\beta}\downarrow & & \downarrow \\ S^{-1}M & \xrightarrow{\ S^{-1}\alpha\ } & \mathrm{Hom}_{S^{-1}R}(\mathrm{Hom}_{S^{-1}R}(M,S^{-1}R),S^{-1}R), \end{array}$$

where the vertical arrows are the mapping induced by the localization by S. If M is torsion-free, then β is injective. Since $S^{-1}R$ is a Gorenstein ring of dimension zero, $S^{-1}\alpha$ is an isomorphism. As a result, it follows that α is injective, hence M is torsionless. □

Theorem 4.2. *Let R be a generically Gorenstein ring. Then the following two conditions are equivalent for $X \in \mathscr{K}(R)$:*

(1) *X is *torsion-free.*

(2) *Each cohomology module $H^i(X^*)$ is a torsion-free R-module for $i \in \mathbb{Z}$.*

Proof. (1) $\Rightarrow$ (2): Before the proof we recall that N^* is torsionless for any finitely generated R-module N. (If $R^m \to N$ is a surjective mapping of R-modules, then we have an injection N^* to a free module $(R^m)^*$.)

By definition $\rho^i_{XR} : H^{-i}(X^*) \to H^i(X)^*$ is injective. Since $H^i(X)$ is a finitely generated R-module, $H^i(X)^*$ is torsionless. This forces $H^{-i}(X^*)$ to also be torsionless, and hence torsion-free.

(2) $\Rightarrow$ (1): Let $S = R \setminus \bigcup_{\mathfrak{p} \in \mathrm{Ass}(R)} \mathfrak{p}$ as in Lemma 4.1. Now let $f : X \to R[i]$ be a chain map with $i \in \mathbb{Z}$ and assume $H(f) = 0$. We want to show that $f = 0$ as an element of $H^i(X^*)$.

Note that $S^{-1}f : S^{-1}X \to S^{-1}R[i]$ is a chain map with $H(S^{-1}f) = 0$. Since $S^{-1}R$ is a Gorenstein ring of dimension zero, we have from Corollary 3.5 that $S^{-1}X$ is *torsion-free as a complex over $S^{-1}R$, hence $S^{-1}f = 0$ in $\mathscr{K}(S^{-1}R)$. This means that $S^{-1}f = 0$ as an element of $H(\mathrm{Hom}_{S^{-1}R}(S^{-1}X, S^{-1}R[i]))$. Since each term of X is a finitely generated R-module, we note that there is a natural isomorphism $\mathrm{Hom}_{S^{-1}R}(S^{-1}X, S^{-1}R[i]) \cong S^{-1}\mathrm{Hom}_R(X, R[i])$, hence

$$H(\mathrm{Hom}_{S^{-1}R}(S^{-1}X, S^{-1}R[i])) \cong S^{-1}H(\mathrm{Hom}_R(X, R[i])) = S^{-1}H^i(X^*).$$

This shows that there is an element $s \in S$ with $sf = 0$ as an element of $H^i(X^*)$. Since we assumed that $H^i(X^*)$ is a torsion-free R-module, we must have $f = 0$ as an element of $H^i(X^*)$. □

Remark 4.3. The implication (1) $\Rightarrow$ (2) in the theorem is generally true without the assumption of generic Gorensteinness. But it is not the case for (2) $\Rightarrow$ (1).

For example, let $(R, \mathfrak{m}, k)$ be a local ring with $\dim R > 0$ and $\mathrm{depth} R = 0$. Note in this case that every k-vector space is torsionless, hence torsion-free, as an R-module, since k is isomorphic to a submodule in R. Now let X be an R-free resolution of k. Then it follows that $H^i(X^*) \cong \mathrm{Ext}^i_R(k, R)$ is a torsion-free R-module for each i, hence X satisfies the condition (2). On the other hand, we note that $H^i(X)^* \neq 0$ only if $i = 0$. Hence the condition (1) forces $\mathrm{Ext}^i_R(k, R) = 0$ for all $i > 0$, which is an equivalent condition for R to be a Gorenstein ring of dimension zero. Therefore X does not satisfy the condition (1).

Note that a finitely generated module M over a commutative Noetherian ring R is said to be reflexive if the natural mapping $M \to M^{**}$ is an isomorphism.

Recall that a commutative Noetherian ring R is said to be **Gorenstein in depth one** if each $R_\mathfrak{p}$ is a Gorenstein ring for all the prime ideals $\mathfrak{p}$ satisfying $\mathrm{depth} R_\mathfrak{p} \leqq 1$.

First we remark the following (perhaps well-known) lemma.

Lemma 4.4. *Assume that R is Gorenstein in depth one.*

(1) *If M is a finitely generated R-module, then M^* is a reflexive R-module.*

(2) *Let $M \subseteq N$ be a submodule of a finitely generated R-module which is equal in depth one, i.e. $M_\mathfrak{p} = N_\mathfrak{p}$ if $\mathrm{depth} R_\mathfrak{p} \leqq 1$. Furthermore assume that both M and N are reflexive. Then $M = N$.*

Proof. (1) Since M^* is a torsionless module, the natural mapping $\alpha : M^* \to M^{***}$ is injective. Set C to be the cokernel of this map, i.e. $C = \mathrm{Cok}(\alpha)$. Then, by the assumption, we have $C_\mathfrak{p} = 0$ if $\mathrm{depth} R_\mathfrak{p} \leqq 1$. (Note that $M^*_\mathfrak{p}$ are torsion-free, hence MCM's over Gorenstein rings $R_\mathfrak{p}$ for those $\mathfrak{p}$, hence $\alpha_\mathfrak{p}$ are isomorphisms.) To prove $C = 0$, let us assume that $C \neq 0$ and take a minimal prime ideal $\mathfrak{p}$ in $\mathrm{Supp}(C)$. Then, by the above, we must have $\mathrm{depth} R_\mathfrak{p} \geqq 2$. Note that there is an exact sequence of $R_\mathfrak{p}$-modules $0 \to M^*_\mathfrak{p} \to M^{***}_\mathfrak{p} \to C_\mathfrak{p} \to 0$, where $C_\mathfrak{p}$ is a non-zero $R_\mathfrak{p}$-module of finite length. Remark here that both $M^*_\mathfrak{p}$ and $M^{***}_\mathfrak{p}$ are second syzygy modules over $R_\mathfrak{p}$. Since $\mathrm{depth} R_\mathfrak{p} \geqq 2$, it follows that such second syzygy modules have depth at least two. Noticing that $\mathrm{depth}\, C_\mathfrak{p} = 0$, we see that this contradicts the depth lemma (see [4, Proposition 1.2.9]).

(2) Setting $C = N/M$, we want to show $C = 0$. By the assumption, if $\mathrm{depth} R_\mathfrak{p} \leqq 1$, then $C_\mathfrak{p} = 0$. Thus every prime $\mathfrak{p}$ in $\mathrm{Supp}(C)$ satisfies $\mathrm{depth} R_\mathfrak{p} \geqq 2$. Assuming $C \neq 0$, we take a minimal prime ideal in $\mathrm{Supp}(C)$. Then there is an exact sequence of $R_\mathfrak{p}$-modules $0 \to M_\mathfrak{p} \to N_\mathfrak{p} \to C_\mathfrak{p} \to 0$, where $C_\mathfrak{p}$ is of finite length. Since $M_\mathfrak{p}$ (resp. $N_\mathfrak{p}$) is a reflexive $R_\mathfrak{p}$-module, the depth of $M_\mathfrak{p}$ (resp. $N_\mathfrak{p}$) is at least two. This contradicts the depth lemma again. □

Theorem 4.5. *Suppose that R is Gorenstein in depth one and let X be a complex in $\mathscr{K}(R)$. Then the following two conditions are equivalent:*

(1) *X is *reflexive.*

(2) *Each cohomology module $H^i(X^*)$ is a reflexive R-module for $i \in \mathbb{Z}$.*

Proof. (1) ⇒ (2): Since X is *reflexive, we have an isomorphism $\rho^i_{XR} : H^{-i}(X^*) \to H^i(X)^*$ for all $i \in \mathbb{Z}$. Note that each $H^i(X)$ is a finitely generated R-module. It thus follows from Lemma 4.4(1) that $H^i(X)^*$, hence $H^{-i}(X^*)$ as well, is a reflexive R-module.

(2) ⇒ (1): We want to show that the natural mapping $\rho^i_{XR} : H^{-i}(X^*) \to H^i(X)^*$ is an isomorphism for each $i \in Z$. We know, from Theorem 4.2, that X is *torsion-free, hence all ρ^i_{XR} are injective. Thus, applying Lemma 4.4(2), we have only to show that $(\rho^i_{XR})_{\mathfrak{p}}$ are isomorphisms for prime ideals $\mathfrak{p}$ with $\operatorname{depth} R_{\mathfrak{p}} \leqq 1$. Therefore the proof is reduced to the case where the ring R is a Gorenstein local ring of dimension at most one. Henceforth we assume R is such a ring. In this case, we have $\operatorname{Ext}^2_R(C(X), R) = 0$, thus it results from Theorem 2.3 that $\rho^i_{XR} : H^{-i}(X^*) \to H^i(X)^*$ is surjective for each $i \in \mathbb{Z}$. Since we know already that this is injective, each ρ^i_{XR} is an isomorphism. □

5. Split complexes and Add(R)

We note that $\mathscr{K}(R)$ admits finite direct sums, and moreover some kind of infinite direct sums can be possibly taken inside $\mathscr{K}(R)$. For example, let $\{X_j \mid j \in J\}$ be a set of complexes in $\mathscr{K}(R)$ and assume that $X^i = \bigoplus_{j\in J} X^i_j$ is a finitely generated R-module for each $i \in \mathbb{Z}$. In such a case the direct sum $X = \coprod_{j\in J} X_j$ (or the coproduct in $\mathscr{K}(R)$) is well-defined so that its ith component is X^i. Note in this case that the direct sum coincides with the direct product $\prod_{j\in J} X_j$, as we see in the next lemma.

The direct sum $\coprod_{i\in\mathbb{Z}} R[i]$ is one of such typical examples of infinite direct sums, actually it is a complex of the form $\left[\cdots \xrightarrow{0} R \xrightarrow{0} R \xrightarrow{0} R \xrightarrow{0} \cdots \right]$ that belongs to $\mathscr{K}(R)$.

Lemma 5.1. *Let $\{X_j \mid j \in J\}$ be a set of complexes in $\mathscr{K}(R)$. Assume that, for each $i \in \mathbb{Z}$, there is a finite subset $J_i \subseteq J$ such that $X^i_j \neq 0$ only if $j \in J_i$. Then the coproduct $X = \coprod_{j\in J} X_j$ exists in $\mathscr{K}(R)$. Moreover in this case, the coproduct is a product in $\mathscr{K}(R)$, i.e. $X = \prod_{j\in J} X_j$. Hence there is an isomorphism of R-modules*

$$\operatorname{Hom}_{\mathscr{K}(R)}(Y, \coprod_{j\in J} X_j) \cong \prod_{j\in J} \operatorname{Hom}_{\mathscr{K}(R)}(Y, X_j)$$

for all $Y \in \mathscr{K}(R)$.

Proof. Let Mod(R) be the abelian category consisting of all (not necessarily finitely generated) R-modules and we denote by $K(\operatorname{Mod}(R))$ the homotopy category of all complexes over Mod(R). Now regarding $\{X_j \mid j \in J\}$ as an object set in $K(\operatorname{Mod}(R))$, we see that the coproduct X in $K(\operatorname{Mod}(R))$ is given as $X^i = \bigoplus_{j\in J} X^i_j$ with differentials defined diagonally by each $d^i_{X_j}$. Similarly the product in $K(\operatorname{Mod}(R))$ is given as $\prod_{j\in J} X^i_j$. Now the assumption of the lemma assures that each X^i is finitely generated, hence the coproduct X in $K(\operatorname{Mod}(R))$ lies in its full subcategory $\mathscr{K}(R)$. This shows that X is in fact a coproduct in the category $\mathscr{K}(R)$.

Moreover, under the assumption in the lemma we have the equality $\bigoplus_{j\in J} X^i_j = \prod_{j\in J} X^i_j$ as R-modules for all $i \in \mathbb{Z}$. Hence the last half of the lemma follows. □

Definition 5.2. Given an $X \in \mathscr{K}(R)$, we define Add(X) as the smallest additive subcategory of $\mathscr{K}(R)$ containing X that is closed under the shift functor and admits possibly infinite coproducts. Equivalently $\mathrm{Add}(X)$ is the intersection of all the full subcategories $\mathcal{U}$ satisfying the following conditions:

(i) $\mathcal{U}$ is closed under isomorphism and $X \in \mathcal{U}$.

(ii) If $Y \in \mathcal{U}$ then $Y[i] \in \mathcal{U}$ for all $i \in \mathbb{Z}$.

(iii) If Z is a direct summand of $Y \in \mathcal{U}$ then $Z \in \mathcal{U}$.

(iv) Let $\{Y_j \mid j \in J\}$ be a set of objects in $\mathcal{U}$ and assume that the coproduct $\coprod_{j\in J} Y_j$ in $\mathscr{K}(R)$ exists. Then $\coprod_{j\in J} Y_j \in \mathcal{U}$.

(Note that 0 is an object of $\mathcal{U}$ by (iii) and that all null complexes belong to $\mathcal{U}$ by (i).)

In the rest of the paper we are particularly interested in $\mathrm{Add}(R)$, where R is regarded as a complex concentrated in degree 0.

If the complex

$$X = \left[\cdots \xrightarrow{d_X^{-2}} X^{-1} \xrightarrow{d_X^{-1}} X^0 \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} \cdots \right],$$

satisfies the equalities $d_X^i = 0$ for all $i \in \mathbb{Z}$, then X belongs to $\mathrm{Add}(R)$, since X is a direct sum $\coprod_{i\in\mathbb{Z}} X^i[-i]$ with each X^i being a projective R-module. Such a complex X is characterized by the condition that $X \cong H(X)$ in $\mathscr{C}(R)$, where we regard the graded R-module $H(X)$ as a complex with zero differentials.

Recall that a complex $X \in \mathscr{C}(R)$ is called **split** if there is a graded R-module homomorphisms $s : X \to X[-1]$ satisfying $d_X s d_X = d_X$. (Cf. [12, Definition (1.4.1)].) To state the following well-known lemma, we recall the notation $C(X) = \mathrm{Coker}(d_X)$ and $B(X) = \mathrm{Im}(d_X)$, for a complex $X \in \mathscr{C}(R)$, as in Notation 2.1.

Lemma 5.3. *The following conditions are equivalent for $X \in \mathscr{C}(R)$.*

(1) *X is split.*

(2) *There is a direct sum decomposition $X = X' \oplus N$ in $\mathscr{C}(R)$ where $d_{X'} = 0$ and N is a null complex.*

(3) *$C(X) = \bigoplus_{i\in\mathbb{Z}} C^i(X)$ is a projective R-module.*

(4) *The natural inclusion map $B(X) = \bigoplus_{i\in\mathbb{Z}} B^i(X) \hookrightarrow X = \bigoplus_{i\in\mathbb{Z}} X^i$ is a split monomorphism as graded R-modules.*

Proof. The implications $(2) \Rightarrow (1) \Rightarrow (4) \Rightarrow (3)$ are well-known and easily proved. We have only to show $(3) \Rightarrow (2)$.

If $C(X)$ is projective, then the natural exact sequences of graded R-modules

$$0 \longrightarrow B(X) \longrightarrow X \longrightarrow C(X) \longrightarrow 0, \qquad 0 \longrightarrow H(X) \longrightarrow C(X) \longrightarrow B(X)[1] \longrightarrow 0$$

are splitting. Therefore each X^i decomposes to $X_0^i \oplus X_1^i \oplus X_2^i$ where $X_0^i \cong H^i(X)$ and $X_1^i \cong B^i(X)$, $X_2^i \cong B^{i+1}(X)$ for $i \in \mathbb{Z}$, and the differential map d_X^i yields an isomorphism $X_2^i \to X_1^{i+1}$, while it is zero on $X_0^i \oplus X_1^i$. Thus, part $X_1 \oplus X_2$ of X defines a null subcomplex N. Therefore, setting $X' = X_0$ with zero differentials, we have a direct sum decomposition $X = X' \oplus N$. $\square$

As a result of the equivalence (1) $\Leftrightarrow$ (2) in the lemma, we see that all the split complexes in $\mathscr{K}(R)$ are belonging to $\mathrm{Add}(R)$. We can show that the uniqueness of the direct sum decomposition in the meaning of (2) in the lemma holds for a split complex.

Lemma 5.4. *Let X be a split complex belonging to $\mathscr{C}(R)$. Assume there are decompositions $X = X_1 \oplus N_1 = X_2 \oplus N_2$ where $d_{X_i} = 0$ and N_i is a null complex for $i = 1, 2$. Then we have isomorphisms $X_1 \cong X_2$, $N_1 \cong N_2$ in $\mathscr{C}(R)$.*

Proof. Write the natural injection $X_1 \hookrightarrow X = X_2 \oplus N_2$ as $\binom{a}{b}$ where $a : X_1 \to X_2$ and $b : X_1 \to N_2$. Similarly write the natural projection $X_2 \oplus N_2 = X \twoheadrightarrow X_1$ as (c, d) with $c : X_2 \to X_1$ and $d : N_2 \to X_1$. Then we have $1_{X_1} = ca + db$. Since the morphism db factors through a null complex, it is null homotopic. Hence it follows from the next remark that $db = 0$ as a morphism in $\mathscr{C}(R)$. Thus $ca = 1_{X_1}$. In the same way as this one can show $ac = 1_{X_2}$. Hence $a : X_1 \to X_2$ is an isomorphism in $\mathscr{C}(R)$.

To show $N_1 \cong N_2$ in $\mathscr{C}(R)$ we remark that, for a null complex N, we have $Z(N) \cong N/Z(N)[-1]$ as graded R-modules and N is isomorphic to the mapping cone of the identity mapping on $N/Z(N)$. Since $Z(X_i) = X_i$ for $i = 1, 2$, we have $X/Z(X) = (X_i \oplus N_i)/(Z(X_i) \oplus Z(N_i)) \cong N_i/Z(N_i)$. Therefore $N_1/Z(N_1) \cong N_2/Z(N_2)$ as graded R-modules. Since both N_1 and N_2 are null complexes, we have an isomorphism $N_1 \cong N_2$ in $\mathscr{C}(R)$, as remarked above. $\square$

Remark 5.5. Let $f : X \to Y$ be a morphism in $\mathscr{C}(R)$, where we assume that $d_X = d_Y = 0$. If f is null homotopic, then $f = 0$ in $\mathscr{C}(R)$. In fact, this follows from that $f = d_Y h - h d_X = 0$ for a homotopy h.

By a similar proof to the lemma above we can also show the following lemma.

Lemma 5.6. *Let X and Y be complexes in $\mathscr{C}(R)$ such that $d_X = 0$. If X is a direct summand of Y in $\mathscr{K}(R)$, then it is also a direct summand of Y in $\mathscr{C}(R)$.*

Proof. Assume there are morphisms $f : X \to Y$ and $g : Y \to X$ in $\mathscr{C}(R)$ such that gf is chain homotopic to the identity morphism 1_X on X. Then it follows from the remark above that $1_X - gf = 0$ as a morphism in $\mathscr{C}(R)$. $\square$

Proposition 5.7. *Let $\{X_j \mid j \in J\}$ be a set of complexes in $\mathscr{C}(R)$ such that $d_{X_j} = 0$ for all $j \in J$. Assume that the coproduct $\coprod_{j \in J} X_j$ in $\mathscr{K}(R)$ exists. Then, for any $i \in \mathbb{Z}$, there is a finite subset $J_i \subseteq J$ such that $X_j^i \neq 0$ only if $j \in J_i$. In this case, the coproduct is an ordinary direct sum of complexes. Hence $\coprod_{j \in J} X_j$ has zero differentials, and it is a split complex as well.*

Proof. Set $P = \coprod_{j \in J} X_j$. By definition we have an isomorphism

$$\mathrm{Hom}_{\mathscr{K}(R)}(P, -) \cong \prod_{j \in J} \mathrm{Hom}_{\mathscr{K}(R)}(X_j, -) \cong \mathrm{Hom}_{K(\mathrm{Mod}(R))}(\bigoplus_{j \in J} X_j, -)|_{\mathscr{K}(R)}$$

as functors on $\mathscr{K}(R)$, where $\bigoplus_{j \in J} X_j$ denotes the coproduct in $K(\mathrm{Mod}(R))$. Therefore there is a morphism $\bigoplus_{j \in J} X_j \to P$ in $K(\mathrm{Mod}(R))$, by which any finite direct sums

$\bigoplus_{k=1}^{r} X_{j_k}$ are direct summands of P in the category $\mathscr{K}(R)$. Then it follows from the previous lemma that any such finite direct sums $\bigoplus_{k=1}^{r} X_{j_k}$ are direct summands of P in the category $\mathscr{C}(R)$. In particular, for each $i \in \mathbb{Z}$, any finite direct sum $\bigoplus_{k=1}^{r} X_{j_k}^i$ of R-modules is a direct summand of P^i. However, since each P^i $(i \in \mathbb{Z})$ is finitely generated R-module, one can find a finite set $J_i \subseteq J$ such that $P^i = \bigoplus_{j \in J_i} X_j^i$ and $X_j^i = 0$ for $j \notin J_i$. □

Now we are able to state a main result of this section.

Theorem 5.8. *The following conditions are equivalent for $X \in \mathscr{K}(R)$.*

(1) *X belongs to* $\mathrm{Add}(R)$.

(2) *X is a split complex.*

(3) *The natural mapping*

$$H : \mathrm{Hom}_{\mathscr{K}(R)}(X, Y) \longrightarrow \mathrm{Hom}_{\mathrm{graded} R\mathrm{-mod}}(H(X), H(Y))$$

which sends f to $H(f)$ is injective for all $Y \in \mathscr{K}(R)$.

(4) *The natural mapping H in the condition* (3) *is bijective for all $Y \in \mathscr{K}(R)$.*

Proof. We have shown the implication (2) ⇒ (1) in Lemma 5.3.

(1) ⇒ (2): Let $\mathcal{U}$ be a subcategory of $\mathscr{K}(R)$ consisting of all split complexes. Note that $R \in \mathcal{U}$ and that $\mathcal{U}$ is closed under shift functor, and taking direct summands. If we prove that $\mathcal{U}$ is closed under taking coproducts in $\mathscr{K}(R)$, then $\mathrm{Add}(R) \subseteq \mathcal{U}$ by Definition 5.2 and the proof will be finished.

Let $\{X_j \mid j \in J\}$ be a set of complexes in $\mathcal{U}$. By Lemma 5.4 each X_j is uniquely decomposed into $X_j' \oplus N_j$ with $d_{X_j'} = 0$ and null N_j. Since $X_j \cong X_j'$ in $\mathscr{K}(R)$, replacing X_j with X_j' we may assume $d_{X_j} = 0$ for all $j \in J$. If the coproduct $\coprod_{j \in J} X_j$ exists in $\mathscr{K}(R)$, then it follows from the previous proposition it is split again hence belongs to $\mathcal{U}$.

(2) ⇒ (4): As in the proof above we may assume that $d_X = 0$, hence $X = H(X)$.

Let $f : X \to Y$ be a morphism in $\mathscr{K}(R)$ and assume that $H(f) = 0$. Then the image of f^i is contained in the coboundary $B^i(Y)$ for $i \in \mathbb{Z}$. Since X^i is a projective module, there is an $h^i : X^i \to Y^{i-1}$ with $f^i = d_Y^{i-1} \cdot h^i$. Thus $\{h^i \mid i \in \mathbb{Z}\}$ gives a homotopy, and we have $f = 0$ as a morphism in $\mathscr{K}(R)$.

To show the surjectivity of H, let $a : H(X) \to H(Y)$ be a graded R-module homomorphism. Then each $a^i : H^i(X) = X^i \to H^i(Y)$ is lifted to an R-module mapping from X^i to the cocycle module $Z^i(Y)$. These lifted maps define a chain map $f : X \to Y$ with $H(f) = a$.

(4) ⇒ (3): Obvious.

(3) ⇒ (2): Let M be a finitely generated R-module and let $P \in \mathscr{K}(R)$ be a projective resolution of M. Then it is clear that $\mathrm{Hom}_{\mathscr{K}(R)}(X[i], P) = H^{-i}(\mathrm{Hom}_R(X, M))$, hence the mapping defined by taking cohomology modules $H : \mathrm{Hom}_{\mathscr{K}(R)}(X[i], P) \to \mathrm{Hom}_{\mathrm{graded} R\mathrm{-mod}}(H(X[i]), H(P)) = \mathrm{Hom}_R(H^i(X), M)$ is just the same as $\rho^i_{X,M}$ defined

in Definition 2.2. Thus the condition (3) implies that $\rho^i_{X,M}$ is injective for all $i \in \mathbb{Z}$ and for all $M \in \mathrm{mod}(R)$. It then follows from Theorem 2.3 that $\mathrm{Ext}^1_R(C(X), M) = 0$ for any $M \in \mathrm{mod}(R)$, and therefore $C(X)$ is a projective R-module. Thus X is split by Lemma 5.3. $\square$

As a result of Theorem 5.8, every complex F in $\mathrm{Add}(R)$ is decomposed as $F = \coprod_{j \in \mathbb{Z}} H^j(F)[-j]$ where $H^j(F) \in \mathrm{proj}(R)$ for all $j \in \mathbb{Z}$. Note that $F^* = \coprod_{j \in \mathbb{Z}} H^j(F)^*[j]$ in this case. Moreover it is easy to see that every complex in $\mathrm{Add}(R)$ is *reflexive.

Proposition 5.9. *Let $X, F \in \mathscr{K}(R)$. Assume that F belongs to $\mathrm{Add}(R)$ and that X is *torsion-free (resp. *reflexive). Then the mapping*

$$H \ : \ \mathrm{Hom}_{\mathscr{K}(R)}(X, F) \longrightarrow \mathrm{Hom}_{\mathrm{graded}\, R-\mathrm{mod}}(H(X), H(F)) \ ; \ f \mapsto H(f)$$

is injective (resp. bijective).

Proof. We may take F as it satisfies $d_F = 0$, hence $F = H(F)$. Then, as remarked above, $F = \coprod_{j \in \mathbb{Z}} F^j[-j]$ with $F^j \in \mathrm{proj}(R)$ and this coproduct is also a product. Therefore,

$$\begin{aligned} &\mathrm{Hom}_{\mathscr{K}(R)}(X, F) = \textstyle\prod_{j \in \mathbb{Z}} \mathrm{Hom}_{\mathscr{K}(R)}(X, F^j[-j]), \quad \text{and} \\ &\mathrm{Hom}_{\mathrm{graded}\, R-\mathrm{mod}}(H(X), H(F)) = \textstyle\prod_{j \in \mathbb{Z}} (H^j(X), F^j[-j]). \end{aligned}$$

According as X is *torsion-free or *reflexive, we have that $H : \mathrm{Hom}_{\mathscr{K}(R)}(X, F^j[i]) \longrightarrow \mathrm{Hom}_{\mathrm{graded}\, R-\mathrm{mod}}(H(X), F^j[i])$ is injective or bijective for each $i, j \in \mathbb{Z}$. The proposition follows from this observation. $\square$

The following theorem is one of the crucial results on *torsion-free complexes, on which the proof of the main Theorem 1.1 will deeply rely. See Sections 10 and 12.

Theorem 5.10. *Assume that $X \in \mathscr{K}(R)$ is *torsion-free and that $F \in \mathrm{Add}(R)$. Let $f \in \mathrm{Hom}_{\mathscr{K}(R)}(X, F)$. Setting $S = R \backslash \bigcup_{\mathfrak{p} \in \mathrm{Ass}(R)} \mathfrak{p}$, if $S^{-1}f = 0$ as a morphism $S^{-1}X \to S^{-1}F$ in $\mathscr{K}(S^{-1}R)$, then we have that $f = 0$ as a morphism in $\mathscr{K}(R)$.*

Proof. If $S^{-1}f = 0$ then $H(S^{-1}f) = 0$ as an $S^{-1}R$-module homomorphism $H(S^{-1}X) \to H(S^{-1}F)$. Thus we see that $S^{-1}H(f) = 0$ as a mapping $S^{-1}H(X) \to S^{-1}H(F)$. Since $H(F)$ is a projective R-module, any elements of S act on $H(F)$ as non zero divisors. It thus follows that $H(f) = 0$ as a mapping $H(X) \to H(F)$. Then from Proposition 5.9 we have $f = 0$. $\square$

Corollary 5.11. *If $X \in \mathscr{K}(R)$ is *torsion-free and $F \in \mathrm{Add}(R)$, then $\mathrm{Hom}_{\mathscr{K}(R)}(X, F)$ is a torsion-free R-module.*

Proof. There is a commutative diagram of R-modules

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{K}(R)}(X, F) & \xrightarrow{\ \alpha\ } & S^{-1}\mathrm{Hom}_{\mathscr{K}(R)}(X, F) \\ & \searrow^{\gamma} & \downarrow{\scriptstyle \beta} \\ & & \mathrm{Hom}_{\mathscr{K}(S^{-1}R)}(S^{-1}X, S^{-1}F), \end{array}$$

where α is a localization mapping by S and γ is a natural mapping that sends f to $S^{-1}f$. Note that $\beta(f/s) = \gamma(f)/s$ for $f \in \mathrm{Hom}_{\mathscr{K}(R)}(X, F)$ and $s \in S$. We have shown in Theorem 5.10 that γ is injective. Thus α is also injective, and hence $\mathrm{Hom}_{\mathscr{K}(R)}(X, F)$ is a torsion-free R-module. □

Remark 5.12. In the proof of the corollary, we should note that the natural mapping

$$\beta : S^{-1}\mathrm{Hom}_{\mathscr{K}(R)}(X, F) \longrightarrow \mathrm{Hom}_{\mathscr{K}(S^{-1}R)}(S^{-1}X, S^{-1}F)$$

is not necessarily an isomorphism. For example, setting $X = F = \coprod_{i\in\mathbb{Z}} R[-i]$, we have $\mathrm{Hom}_{\mathscr{K}(R)}(X, F) = \prod_{i\in\mathbb{Z}} R$ and $\mathrm{Hom}_{\mathscr{K}(S^{-1}R)}(S^{-1}X, S^{-1}F) = \prod_{i\in\mathbb{Z}} S^{-1}R$.

6. The stable category of $\mathscr{K}(R)$

The main objective of this paper is to consider the nature of complexes in $\mathscr{K}(R)$ up to $\mathrm{Add}(R)$-summands, which we should call the stable theory after the paper [2].

Definition 6.1. We denote by $\underline{\mathscr{K}(R)}$ the factor category $\mathscr{K}(R)$ modulo the subcategory $\mathrm{Add}(R)$:

$$\underline{\mathscr{K}(R)} = \mathscr{K}(R)/\mathrm{Add}(R)$$

We call $\underline{\mathscr{K}(R)}$ the **stable category** of $\mathscr{K}(R)$.

The objects of $\underline{\mathscr{K}(R)}$ are the same as $\mathscr{K}(R)$, while the morphism set is given by

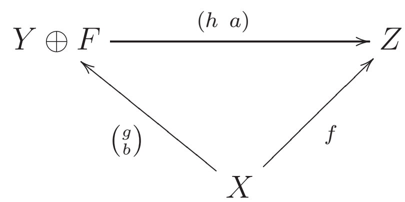
$$\mathrm{Hom}_{\underline{\mathscr{K}(R)}}(X, Y) = \mathrm{Hom}_{\mathscr{K}(R)}(X, Y)/\mathrm{Add}(R)(X, Y),$$

for $X, Y \in \underline{\mathscr{K}(R)}$, where $\mathrm{Add}(R)(X, Y)$ is the R-submodule of $\mathrm{Hom}_{\mathscr{K}(R)}(X, Y)$ consisting of all morphisms factoring through objects of $\mathrm{Add}(R)$. The object sets of $\mathscr{K}(R)$ and $\underline{\mathscr{K}(R)}$ are identical, but for an object $X \in \mathscr{K}(R)$, to discriminate it with an object in $\underline{\mathscr{K}(R)}$, we often write $\underline{X}$ for the corresponding object in $\underline{\mathscr{K}(R)}$. Similarly we denote by $\underline{f}$ the corresponding morphism in $\underline{\mathscr{K}(R)}$ for a given f in $\mathscr{K}(R)$.

Since $\mathrm{Add}(R)$ is stable under the action of shift functor in $\mathscr{K}(R)$, it should be noted that $\underline{\mathscr{K}(R)}$ admits the shift functor so that $\underline{X[1]} = \underline{X}[1]$ for $X \in \underline{\mathscr{K}(R)}$. However $\underline{\mathscr{K}(R)}$ is not a triangulated category, but merely an additive R-linear category with the shift functor that is an auto-functor on it. ($\underline{\mathscr{K}(R)}$ is not triangulated, by which we mean that there is no triangle structure on $\underline{\mathscr{K}(R)}$ that makes the natural functor $\mathscr{K}(R) \to \underline{\mathscr{K}(R)}$ a triangle functor. This is true, since $\mathrm{Add}(R)$ is not closed under triangles in $\mathscr{K}(R)$.)

First of all we remark on the commutativity of a diagram in $\underline{\mathscr{K}(R)}$.

Lemma 6.2. *Let $f : X \to Z$, $g : X \to Y$, $h : Y \to Z$. Then $\underline{f} = \underline{h} \cdot \underline{g}$ in $\underline{\mathscr{K}(R)}$ if and only if there is a commutative diagram in $\mathscr{K}(R)$ of the following form:*



where $F \in \mathrm{Add}(R)$.

Proof. If $f - hg$ factors through $F \in \mathrm{Add}(R)$, then there are $a : F \to Z$ and $b : X \to F$ that satisfy the equality $f = hg + ab$. The converse is trivial since $\underline{a} = \underline{b} = 0$. □

Note from this lemma that $\underline{X} = 0$ for $X \in \mathscr{K}(R)$ if and only if $X \in \mathrm{Add}(R)$. In fact if $1_{\underline{X}} = 0$, then setting $X = Z$, $Y = 0$ and $f = 1_X$ in the lemma, we see that X is a direct summand of $F \in \mathrm{Add}(R)$ and hence $X \in \mathrm{Add}(R)$. More generally we should note the following corollary holds.

Corollary 6.3. *Let $X, Y \in \mathscr{K}(R)$. Then $\underline{X} \cong \underline{Y}$ in $\underline{\mathscr{K}(R)}$ if and only if $X \oplus F \cong Y \oplus F'$ in $\mathscr{K}(R)$ for some $F, F' \in \mathrm{Add}(R)$.*

Proof. If $\underline{g} : \underline{X} \to \underline{Y}$ is an isomorphism whose inverse morphism is $\underline{h}$, then it follows from Lemma 6.2 that X is a direct summand of $Y \oplus F$ in $\mathscr{K}(R)$ for some $F \in \mathrm{Add}(R)$. Therefore there exists an isomorphism $Y \oplus F \to X \oplus F'$ in which the restricted map $Y \to X$ is given by h. We have to show that $F' \in \mathrm{Add}(R)$. Since $\underline{F} = 0$, we have an isomorphism $\underline{Y} \to \underline{X} \oplus \underline{F'}$ in which $\underline{h} : \underline{Y} \to \underline{X}$ is also an isomorphism. Then it is an easy exercise to show $\underline{F'} = 0$ in $\underline{\mathscr{K}(R)}$, hence $F' \in \mathrm{Add}(R)$. □

Remark 6.4. Recall from Theorem 5.8 that $X \in \mathscr{K}(R)$ belongs to $\mathrm{Add}(R)$ if and only if X is a split complex. Hence, setting $\mathcal{S}$ to be the full subcategory of $\mathscr{C}(R)$ that consists of all split complexes, we can also describe the stable category as $\underline{\mathscr{K}(R)} = \mathscr{C}(R)/\mathcal{S}$. Therefore one can also prove that $\underline{X} \cong \underline{Y}$ in $\underline{\mathscr{K}(R)}$ if and only if $X \oplus T \cong Y \oplus T'$ in $\mathscr{C}(R)$ for some $T, T' \in \mathcal{S}$.

Definition 6.5. Let $f : X \to Y$ be a morphism in $\mathscr{K}(R)$. We say that f is **cohomologically surjective** if the cohomology mapping $H(f) : H(X) \to H(Y)$ is surjective.

We also define the complex $Cone(f) \in \mathscr{K}(R)$ by the triangle

$$Cone(f)[-1] \longrightarrow X \xrightarrow{\ f\ } Y \longrightarrow Cone(f)$$

in $\mathscr{K}(R)$, which is actually the mapping cone of the chain map f.

In general, for given morphisms $f, g : X \to Y$ in $\mathscr{K}(R)$, that $\underline{f} = \underline{g}$ in $\underline{\mathscr{K}(R)}$ does not mean $\underline{Cone(f)} \cong \underline{Cone(g)}$ in $\underline{\mathscr{K}(R)}$. But so does it if they are cohomologically surjective.

Theorem 6.6. *Let $f : X \to Y$ and $f' : X' \to Y$ be morphisms in $\mathscr{K}(R)$. Assume that both f and f' are cohomologically surjective. Further assume that $\underline{X} \cong \underline{X'}$ in $\underline{\mathscr{K}(R)}$ and that $\underline{f}$ corresponds to $\underline{f'}$ under the isomorphism $\mathrm{Hom}_{\underline{\mathscr{K}(R)}}(\underline{X}, \underline{Y}) \cong \mathrm{Hom}_{\underline{\mathscr{K}(R)}}(\underline{X'}, \underline{Y})$. Then we have an isomorphism $\underline{Cone(f)} \cong \underline{Cone(f')}$ in $\underline{\mathscr{K}(R)}$.*

Proof. As the first step of the proof we prove the following isomorphism:

$$\underline{Cone(f)} \cong \underline{Cone(f\ a)} \quad \text{for any} \quad F \in \mathrm{Add}(R) \text{ and } (f\ a) : X \oplus F \to Y. \tag{6.1}$$

In fact, there is a commutative diagram in $\mathscr{K}(R)$ whose rows and columns are triangles:

$$\begin{array}{ccccccc}
 & & F[-1] & = & F[-1] & & \\
 & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle 0} & & \\
Y[-1] & \longrightarrow & Cone(f)[-1] & \xrightarrow{v} & X & \xrightarrow{f} & Y \\
\| & & \downarrow & & \downarrow{\scriptstyle \binom{1}{0}} & & \| \\
Y[-1] & \longrightarrow & Cone(f\ a)[-1] & \longrightarrow & X \oplus F & \xrightarrow{(f\ a)} & Y \\
 & & \downarrow & & \downarrow{\scriptstyle (0\ 1)} & & \\
 & & F & = & F. & &
\end{array}$$

Since $H(f)$ is surjective, note in this diagram that $H(v)$ is injective. Then that $vu = 0$, and hence $H(v)H(u) = 0$, forces $H(u) = 0$. Thus by Theorem 5.8 we have $u = 0$, which shows an isomorphism $Cone(f\ a)[-1] \cong Cone(f)[-1] \oplus F$, and hence (6.1) is proved.

As the second step of the proof, we prove the theorem in the case of $X = X'$. In this case we have $f' = f + ab$ for $a : F \to Y$ and $b : X \to F$ with $F \in \mathrm{Add}(R)$, by virtue of Lemma 6.1. Then there is a commutative diagram in $\mathscr{K}(R)$

$$\begin{array}{ccc}
X \oplus F & \xrightarrow{(f'\ a)} & Y \\
{\scriptstyle \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}}\downarrow & & \| \\
X \oplus F & \xrightarrow{(f\ a)} & Y.
\end{array}$$

Since the left vertical arrow is an isomorphism, we have $Cone(f\ a) \cong Cone(f'\ a)$ in $\mathscr{K}(R)$, hence $\underline{Cone(f)} \cong \underline{Cone(f')}$ by using (6.1).

Now consider the general case of the theorem. Since $\underline{X} \cong \underline{X'}$, there is an isomorphism $g : X \oplus F \to X' \oplus F'$ for some $F, F' \in \mathrm{Add}(R)$, and by the assumption we must have $\underline{f} = \underline{f'} \cdot \underline{g}$. Consider the morphisms $(f\ 0) : X \oplus F \to Y$ and $(f'\ 0) : X' \oplus F' \to Y$, and we note that they are cohomologically surjective. On the other hand, since $\underline{F} = \underline{F'} = 0$, we have equalities

$$\underline{(f\ \ 0)} = \underline{f} = \underline{f'} \cdot \underline{g} = \underline{(f'\ \ 0) \cdot g}.$$

Thus it follows from the second step of this proof that $\underline{Cone(f\ 0)} \cong \underline{Cone((f'\ 0)\cdot g)}$. Note here that $Cone((f'\ 0)\cdot g) \cong Cone(f'\ 0)$ in $\mathscr{K}(R)$, since g is an isomorphism in $\mathscr{K}(R)$. Hence the isomorphism $\underline{Cone(f)} \cong \underline{Cone(f')}$ follows from (6.1). □

7. Add(R)-approximations

We are able to show that the subcategory Add(R) of $\mathscr{K}(R)$ is functorially finite in the sense of Auslander. (Cf. Auslander [1].) For this we begin with recalling the definition of right approximations.

Definition 7.1. Let $X \in \mathscr{K}(R)$. A morphism $p : F \to X$ in $\mathscr{K}(R)$ is called a **right** Add(R)**-approximation** of X if $F \in \mathrm{Add}(R)$ and $\mathrm{Hom}_{\mathscr{K}(R)}(G,p) : \mathrm{Hom}_{\mathscr{K}(R)}(G,F) \to \mathrm{Hom}_{\mathscr{K}(R)}(G,X)$ is surjective for any $G \in \mathrm{Add}(R)$.

We should remark that the shift functor preserves the right Add(R)-approximation property, i.e. $p : F \to X$ is a right Add(R)-approximation if and only if so is $p[n] : F[n] \to X[n]$ for any $n \in \mathbb{Z}$.

Lemma 7.2. *Let $X \in \mathscr{K}(R)$ and $F \in \mathrm{Add}(R)$. Then a morphism $p : F \to X$ in $\mathscr{K}(R)$ is a right* Add(R)*-approximation if and only if p is cohomologically surjective. In particular, there always exists a right* Add(R)*-approximation of X for any $X \in \mathscr{K}(R)$.*

Proof. If $p : F \to X$ is a right Add(R)-approximation then $H^i(p) = \mathrm{Hom}_{\mathscr{K}(R)}(R[-i],p)$ is surjective, since $R[-i] \in \mathrm{Add}(R)$ for $i \in \mathbb{Z}$.

Conversely assume that $H(p)$ is surjective, and let $g : G \to X$ be a morphism in $\mathscr{K}(R)$ with $G \in \mathrm{Add}(R)$. Then $H(g) : H(G) \to H(X)$ factors through $H(p)$, since $H(G)$ is a graded projective R-module :

$$\begin{array}{ccc} H(G) & & \\ {\scriptstyle\alpha}\downarrow & \searrow^{H(g)} & \\ H(F) & \xrightarrow{H(p)} & H(X) \end{array}$$

Then, by Theorem 5.8, there is a morphism $a : G \to F$ such that $H(a) = \alpha$, and since $H(g) = H(pa)$, we have $g = pa$.

For the existence of right Add(R)-approximation of X, one has only to take a graded projective R-module F which maps surjectively onto $H(X)$. Then it follows from Theorem 5.8 that this mapping is lifted to a chain homomorphism $F \to X$ which is in fact a right Add(R)-approximation of X. □

If $p : F \to X$ is a right Add(R)-approximation, then as we have shown in Theorem 6.6, the mapping cone $Cone(p)$ is uniquely determined as an object of $\underline{\mathscr{K}(R)}$.

Definition 7.3. Let $X \in \mathscr{K}(R)$ and $p : F \to X$ be a right Add(R)-approximation of X. We define $\Omega(X)$ (or simply denoted ΩX) by the equality

$$\Omega(X) = Cone(p)[-1],$$

which is uniquely determined in the stable category $\underline{\mathscr{K}(R)}$ by Theorem 6.6. Actually, Ω yields a functor $\underline{\mathscr{K}(R)} \to \underline{\mathscr{K}(R)}$ as follows:

Let $a : X \to Y$ be a morphism in $\mathscr{K}(R)$. If $p_X : F_X \to X$ and $p_Y : F_Y \to Y$ are right $\mathrm{Add}(R)$-approximation, then, since ap_X factors through p_Y, we have the following commutative diagram, and as a result the morphism $b : \Omega(X) \to \Omega(Y)$ is induced.

$$\begin{array}{ccccccc}
\Omega(X) & \xrightarrow{q_X} & F_X & \xrightarrow{p_X} & X & \xrightarrow{\omega_X} & \Omega(X)[1] \\
\downarrow{\scriptstyle b} & & \downarrow & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b[1]} \\
\Omega(Y) & \xrightarrow{q_Y} & F_Y & \xrightarrow{p_Y} & Y & \xrightarrow{\omega_Y} & \Omega(Y)[1]
\end{array}$$

If a factors through an objects in $\mathrm{Add}(R)$, then it factors through p_Y, i.e. there is $c : X \to F_Y$ such that $a = p_Y c$. Then we have $b[1]\omega_X = \omega_Y a = \omega_Y p_Y c = 0$, hence there is a morphism $e : F_X \to \Omega(Y)$ with $b[1] = e[1]q_X[1]$ or $b = eq_X$. Thus b factors through an object in $\mathrm{Add}(R)$. In such a way we see that the mapping

$$\mathrm{Hom}_{\underline{\mathscr{K}(R)}}(X, Y) \to \mathrm{Hom}_{\underline{\mathscr{K}(R)}}(\Omega(X), \Omega(Y)) \quad ; \ \underline{a} \ \mapsto \underline{b}$$

is well-defined, hence we can define $\Omega(\underline{a}) = \underline{b}$ for morphisms. We call Ω the **syzygy functor** on $\underline{\mathscr{K}(R)}$.

Definition 7.4. Let $X \in \mathscr{K}(R)$. A morphism $q : X \to G$ in $\mathscr{K}(R)$ is called a **left** $\mathrm{Add}(R)$**-approximation** of X if $G \in \mathrm{Add}(R)$ and $\mathrm{Hom}_{\mathscr{K}(R)}(q, F) : \mathrm{Hom}_{\mathscr{K}(R)}(G, F) \to \mathrm{Hom}_{\mathscr{K}(R)}(X, F)$ are surjective mappings for all $F \in \mathrm{Add}(R)$.

Recall that the dual complex $X^* = \mathrm{Hom}_R(X, R)$ is again a complex belonging to $\mathscr{K}(R)$ and $X^{**} \cong X$. Also note that $X^* \in \mathrm{Add}(R)$ if and only if $X \in \mathrm{Add}(R)$.

Lemma 7.5. *Let $X \in \mathscr{K}(R)$ and $G \in \mathrm{Add}(R)$. Then a morphism $q : X \to G$ in $\mathscr{K}(R)$ is a left $\mathrm{Add}(R)$-approximation if and only if the dual $q^* : G^* \to X^*$ is a right $\mathrm{Add}(R)$-approximation, and the latter is equivalent to that q^* is cohomologically surjective. In particular, there exists a left $\mathrm{Add}(R)$-approximation of X for any $X \in \mathscr{K}(R)$.*

Proof. Assume that $q^* : G^* \to X^*$ is a right $\mathrm{Add}(R)$-approximation. Let $a : X \to F$ be a morphism in $\mathscr{K}(R)$ with $F \in \mathrm{Add}(R)$. Then $a^* : F^* \to X^*$ is a morphism in $\mathscr{K}(R)$, hence it factors through q^*. As a result, a factors through q, hence q is a left $\mathrm{Add}(R)$-approximation. The converse is proved similarly. □

Remark 7.6. Suppose we have a commutative diagram in $\mathscr{K}(R)$ with $F, F' \in \mathrm{Add}(R)$:

$$\begin{array}{ccc}
F & \xrightarrow{p} & X \\
\downarrow{\scriptstyle f} & \nearrow{\scriptstyle p'} & \\
F' & &
\end{array}$$

In such a case if p is right $\mathrm{Add}(R)$-approximation, then so is p'. In fact if $H(p) = H(p')H(f)$ is surjective, then so is $H(p')$.

Similarly if a diagram

$$\begin{array}{ccc} X & \xrightarrow{q} & G \\ & \searrow^{q'} & \uparrow \\ & & G' \end{array}$$

is commutative with $G, G' \in \mathrm{Add}(R)$ and if q is a left $\mathrm{Add}(R)$-approximation, then q' is a left $\mathrm{Add}(R)$-approximation as well.

Corollary 7.7. *Assume that R is a Gorenstein ring of dimension zero, and let*

$$Y \xrightarrow{q} F \xrightarrow{p} X \longrightarrow Y[1] \ ,$$

be a triangle in $\mathscr{K}(R)$ where $F \in \mathrm{Add}(R)$. Then, p is a right $\mathrm{Add}(R)$-approximation if and only if q is a left $\mathrm{Add}(R)$-approximation.

Proof. If p is a right $\mathrm{Add}(R)$-approximation then $H(p)$ is a surjective R-module homomorphism. Then $H(p)^*$ is an injective homomorphism. Since R itself is an injective R-module, noting that the equality $H(p^*) = H(p)^*$ holds, we see from the triangle $X^* \xrightarrow{p^*} F^* \xrightarrow{q^*} Y^* \longrightarrow X^*[1]$ that $H(q^*)$ is surjective. Hence q is a left $\mathrm{Add}(R)$-approximation by Lemma 7.5. The converse is proved in a similar manner. □

Definition 7.8. Let $X \in \mathscr{K}(R)$ and $q : X \to G$ be a left $\mathrm{Add}(R)$-approximation of X. Embed q into a triangle $X \xrightarrow{q} G \longrightarrow Z \longrightarrow X[1]$. We denote the resulted Z by $\Sigma(X)$ (or simply ΣX). It follows from Lemma 7.5 that

$$\Sigma(X) = \Omega(X^*)^*,$$

which is uniquely determined as an object in the stable category $\underline{\mathscr{K}(R)}$. Actually, Σ yields a well-defined functor $\underline{\mathscr{K}(R)} \to \underline{\mathscr{K}(R)}$ as in a similar manner to the case of Ω. We call Σ the **cosyzygy functor** on $\underline{\mathscr{K}(R)}$.

Remark 7.9. If R is a Gorenstein ring of dimension zero, then Corollary 7.7 says that Σ is actually the quasi-inverse of Ω as a functor on $\underline{\mathscr{K}(R)}$.

Proposition 7.10. *Let S be a multiplicative closed subset of R. Then the functor $S^{-1} : \mathscr{K}(R) \to \mathscr{K}(S^{-1}R)$ is defined naturally by taking the localization by S.*

(1) *If $p : F \to X$ is a right $\mathrm{Add}(R)$-approximation in $\mathscr{K}(R)$, then $S^{-1}p : S^{-1}F \to S^{-1}X$ is a right $\mathrm{Add}(S^{-1}R)$-approximation in $\mathscr{K}(S^{-1}R)$.*

(2) *If $q : Y \to G$ is a left $\mathrm{Add}(R)$-approximation in $\mathscr{K}(R)$, then $S^{-1}q : S^{-1}Y \to S^{-1}G$ is a left $\mathrm{Add}(S^{-1}R)$-approximation in $\mathscr{K}(S^{-1}R)$.*

(3) *Let $\Omega_{S^{-1}R}$ and $\Sigma_{S^{-1}R}$ be the syzygy and cosyzygy functors on $\underline{\mathscr{K}(S^{-1}R)}$. Then the following squares are commutative:*

$$\begin{array}{ccc} \underline{\mathscr{K}(R)} & \xrightarrow{\Omega} & \underline{\mathscr{K}(R)} \\ {\scriptstyle S^{-1}}\downarrow & & \downarrow{\scriptstyle S^{-1}} \\ \underline{\mathscr{K}(S^{-1}R)} & \xrightarrow{\Omega_{S^{-1}R}} & \underline{\mathscr{K}(S^{-1}R)}, \end{array} \qquad \begin{array}{ccc} \underline{\mathscr{K}(R)} & \xrightarrow{\Sigma} & \underline{\mathscr{K}(R)} \\ {\scriptstyle S^{-1}}\downarrow & & \downarrow{\scriptstyle S^{-1}} \\ \underline{\mathscr{K}(S^{-1}R)} & \xrightarrow{\Sigma_{S^{-1}R}} & \underline{\mathscr{K}(S^{-1}R)}. \end{array}$$

Proof. (1) If p is cohomologically surjective, then so is $S^{-1}p$.
(2) is clear from the fact that $S^{-1}\mathrm{Hom}_R(q,R) \cong \mathrm{Hom}_{S^{-1}R}(S^{-1}q, S^{-1}R)$ and Lemma 7.5.
(3) follows from (1) and (2). □

In general, Σ is not necessarily the quasi-inverse of Ω, but we see that Σ is a left adjoint to Ω.

Theorem 7.11. *As functors from $\underline{\mathscr{K}(R)}$ to itself, (Σ, Ω) is an adjoint pair, i.e. there are functorial isomorphisms*

$$\mathrm{Hom}_{\underline{\mathscr{K}(R)}}(\Sigma X, Y) \cong \mathrm{Hom}_{\underline{\mathscr{K}(R)}}(X, \Omega Y),$$

for all $X, Y \in \underline{\mathscr{K}(R)}$.

Proof. To prove the theorem let $a : \Sigma X \to Y$ be a morphism in $\mathscr{K}(R)$. Then it induces the following commutative diagram:

$$\begin{array}{ccccccc} X & \xrightarrow{q} & G_X & \longrightarrow & \Sigma X & \longrightarrow & X[1] \\ {\scriptstyle b}\downarrow & & \downarrow & & \downarrow{\scriptstyle a} & & \downarrow \\ \Omega Y & \longrightarrow & F_Y & \xrightarrow{p} & Y & \longrightarrow & \Omega Y, \end{array}$$

where p is a right $\mathrm{Add}(R)$-approximation and q is a left $\mathrm{Add}(R)$-approximation. Then, by the same reason in Definition 7.3 above, we see that

$$\mathrm{Hom}_{\underline{\mathscr{K}(R)}}(\Sigma X, Y) \to \mathrm{Hom}_{\underline{\mathscr{K}(R)}}(X, \Omega Y) \quad ; \ \underline{a} \ \mapsto \underline{b}$$

is well-defined. Conversely, given a morphism $b : X \to \Omega Y$, one can easily find an $a : \Sigma X \to Y$ that makes the diagram commutative. It thus gives the inverse to the above mapping:

$$\mathrm{Hom}_{\underline{\mathscr{K}(R)}}(X, \Omega Y) \to \mathrm{Hom}_{\underline{\mathscr{K}(R)}}(\Sigma X, Y) \quad ; \ \underline{b} \ \mapsto \underline{a}$$

□

We should notice that the similar arguments to ours in this section and also the similar content to Theorem 5.8 can be found in those papers of J.D.Christensen [5, Proposition 8.1] and Krause-Kussin [9, Lemma 2.5].

Example 7.12. Let M be a finitely generated R-module and let

$$\cdots \longrightarrow P_n \xrightarrow{u_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{u_1} P_0 \xrightarrow{u_0} M \longrightarrow 0$$

be a projective resolution of M where each P_i are finitely generated. Now, set the complex X to be $\left[\, 0 \longrightarrow P_1 \xrightarrow{u_1} P_0 \longrightarrow 0 \,\right]$. Then one can easily see that

$$\Omega X = \left[\, 0 \longrightarrow P_2 \xrightarrow{u_2} P_1 \longrightarrow 0 \,\right].$$

More generally we have

$$\Omega^n X (= \Omega(\Omega^{n-1} X)) = \left[\, 0 \longrightarrow P_{n+1} \xrightarrow{u_{n+1}} P_n \longrightarrow 0 \,\right],$$

for $n > 0$. On the other hand let

$$\cdots \longrightarrow Q_n \xrightarrow{v_n} Q_{n-1} \longrightarrow \cdots \longrightarrow Q_1 \xrightarrow{v_1} Q_0 \xrightarrow{v_0} M^* \longrightarrow 0$$

be a projective resolution of M^*. Then one can see that

$$\Sigma X = \left[\, 0 \longrightarrow P_0 \xrightarrow{w} Q_0^* \longrightarrow 0 \,\right],$$

where w is the composition $P_0 \xrightarrow{u_0} M \xrightarrow{\text{natural}} M^{**} \xrightarrow{v_0^*} Q_0^*$. For $n > 1$, it can be easily seen that

$$\Sigma^n X = \left[\, 0 \longrightarrow Q_{n-2}^* \xrightarrow{v_{n-1}^*} Q_{n-1}^* \longrightarrow 0 \,\right].$$

See also Example 9.9.

Lemma 7.13. *Let $X \in \mathscr{K}(R)$. Then ΣX is *torsion-free as an object in $\mathscr{K}(R)$.*

Proof. Suppose we are given $f \in H^i((\Sigma X)^*) = \mathrm{Hom}_{\mathscr{K}(R)}(\Sigma X, R[i])$ for some $i \in \mathbb{Z}$ such that $H(f) = 0$. We want to show $f = 0$.

There is a triangle;

$$\begin{array}{ccccccc} G_X & \xrightarrow{a} & \Sigma X & \xrightarrow{b} & X[1] & \xrightarrow{q} & G_X[1] \\ & & \downarrow{\scriptstyle f} & & & & \\ & & R[i], & & & & \end{array}$$

where q is a left $\mathrm{Add}(R)$-approximation. Since $H(fa) = H(f)H(a) = 0$, we have $fa = 0$ by Theorem 5.8. Therefore there is a morphism $c : X[1] \to R[i]$ such that $f = cb$. Since q is a left $\mathrm{Add}(R)$-approximation and since $R[i] \in \mathrm{Add}(R)$, we have $c = eq$ for some $e : G_X[1] \to R[i]$. Thus $f = cb = eqb = 0$ as desired. □

Theorem 7.14. *The following conditions are equivalent for $X \in \mathscr{K}(R)$.*

(1) *X is *torsion-free.*

(2) *There are complexes $Y \in \mathscr{K}(R)$ and $F \in \mathrm{Add}(R)$ such that X is a direct summand of $\Sigma Y \oplus F$ in $\mathscr{K}(R)$.*

(3) *There is an isomorphism* $X \cong \Sigma\Omega X$ *in* $\underline{\mathscr{K}(R)}$.

Proof. The implications (2) $\Rightarrow$ (1) follows from Lemma 7.13, since direct sums (or direct summands) of *torsion-free complexes are *torsion-free as well.

The condition (3) means exactly that $X \oplus F_1 \cong \Sigma\Omega X \oplus F_2$ in $\mathscr{K}(R)$ for some $F_1, F_2 \in \mathrm{Add}(R)$. Hence (3) implies (2).

It remains to prove (1) $\Rightarrow$ (3). Let

$$\begin{array}{ccccccc} \Omega X & \xrightarrow{a} & F & \xrightarrow{p} & X & \longrightarrow & \Omega X[1] \\ \| & & \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle \pi} & & \| \\ \Omega X & \xrightarrow{q} & G & \longrightarrow & \Sigma\Omega X & \longrightarrow & \Omega X[1] \end{array}$$

be a commutative diagram in $\mathscr{K}(R)$ whose rows are triangles in $\mathscr{K}(R)$, where p (resp. q) is a right (resp. left) $\mathrm{Add}(R)$-approximation and the morphism f is induced by the definition of left $\mathrm{Add}(R)$-approximations. Note in this diagram that we can take such diagram in such a way that $H(f)$ is a surjective graded R-module homomorphism. In fact, if necessary, we may replace $q : \Omega X \longrightarrow G$ by $\binom{q}{a} : \Omega X \longrightarrow G \oplus F$. Thus we may assume that $L := Cone(f)[-1]$ belongs to $\mathrm{Add}(R)$. Note also that π is cohomologically surjective, since both f and p are so. Now we have the following triangle in $\mathscr{K}(R)$ by the octahedron axiom:

$$L \longrightarrow \Sigma\Omega X \xrightarrow{\pi} X \xrightarrow{b} L[1].$$

Since $H(\pi)$ is surjective, we see that $H(b) = 0$ by the cohomology long exact sequence. Then, since we are assuming that X is *torsion-free, it follows from Proposition 5.9 that $b = 0$ as a morphism in $\mathscr{K}(R)$. Thus the triangle splits and we have the isomorphism $\Sigma\Omega X \cong X \oplus L$ in $\mathscr{K}(R)$ with $L \in \mathrm{Add}(R)$. $\square$

Remark 7.15. By the adjoint property proved in Theorem 7.11, there is a natural counit morphism $\pi : \Sigma\Omega X \to X$ for any $X \in \underline{\mathscr{K}(R)}$. Lemma 7.14 says that this is actually a right *torsion-free approximation of X.

Lemma 7.16. *Suppose that R is a generically Gorenstein ring. If $X \in \mathscr{K}(R)$ is *torsion-free, then ΣX is *reflexive.*

Proof. We have shown in Lemma 7.13 that ΣX is *torsion-free. To prove that it is *reflexive, let $\alpha : H^n(\Sigma X) \to R$ be a homomorphism of R-modules, where $n \in \mathbb{Z}$. We want to show that there is a morphism $a : \Sigma X \to R[-n]$ in $\mathscr{K}(R)$ satisfying $H(a) = \alpha$. By definition, there is a triangle

$$X \xrightarrow{q} G_X \xrightarrow{p} \Sigma X \xrightarrow{r} X[1],$$

where q is a left $\mathrm{Add}(R)$-approximation. Therefore we have a long exact sequence of cohomology modules;

$$\begin{array}{ccccccc} H^n(X) & \xrightarrow{H(q)} & H^n(G_X) & \xrightarrow{H(p)} & H^n(\Sigma X) & \xrightarrow{H(r)} & H^{n+1}(X) \\ & & & & \downarrow{\scriptstyle \alpha} & & \\ & & & & R & & \end{array}$$

Note that $G_X \in \mathrm{Add}(R)$ is *reflexive, and thus there is a morphism $b : G_X \to R[-n]$ such that $H(b) = \alpha H(p)$. Then it is clear that $H(bq) = \alpha H(p)H(q) = 0$. Since X is *torsion-free, it follows that $bq = 0$. Thus there is a morphism $a : \Sigma X \to R[-n]$ with $b = ap$. Note that $(\alpha - H(a))H(p) = 0$. Let S be the set of all non-zero divisors of R. Since $S^{-1}R$ is a Gorenstein ring of dimension zero,

$$S^{-1}X \xrightarrow{S^{-1}q} S^{-1}G_X \xrightarrow{S^{-1}p} S^{-1}\Sigma X \xrightarrow{S^{-1}\omega} S^{-1}X[1]$$

is a triangle in which $S^{-1}q$ is a left $\mathrm{Add}(S^{-1}R)$-approximation and $S^{-1}p$ is a right approximation in $\mathscr{K}(S^{-1}R)$ by Corollary 7.7. In particular, $S^{-1}H(p) = H(S^{-1}p)$ is a surjective mapping by Lemma 7.2. Since $(\alpha - H(a))H(p) = 0$, we see that $S^{-1}(\alpha - H(a)) = 0$ as an element of $S^{-1}(H(\Sigma X)^*)$. Noting that the R-dual of any finitely generated module is torsion-free, we see that $H(\Sigma X)^*$ is a torsion-free R-module. Consequently we have that $\alpha = H(a)$ as an element of $H^n(\Sigma(X))^*$. □

Combining this lemma with Theorem 7.14 or with Lemma 7.13 we obtain the following theorem.

Theorem 7.17. *Under the assumption that R is generically Gorenstein, $\Sigma^2 X = \Sigma(\Sigma X)$ is always *reflexive for any $X \in \mathscr{K}(R)$.*

Similarly to Theorem 7.14 one can characterize the *reflexivity property for complexes as follows:

Corollary 7.18. *Assume that R is a generically Gorenstein ring. Then the following two conditions for $X \in \mathscr{K}(R)$ are equivalent:*

(1) *X is *reflexive.*
(2) *$\Sigma^2\Omega^2 X \cong X$ in $\underline{\mathscr{K}(R)}$.*

Proof. Theorem 7.17 says that (2) ⇒ (1) holds. Assume X is *reflexive. Then take a right $\mathrm{Add}(R)$-approximation sequence $\Omega X \longrightarrow F_0 \longrightarrow X \longrightarrow \Omega X[1]$, and applying Proposition 3.7(2), we see that ΩX is *torsion-free. Then it follows from Theorem 7.14 that $\Sigma\Omega^2 X = \Sigma\Omega(\Omega X) \cong \Omega X$. Thus applying Σ to the both sides, we have $\Sigma^2\Omega^2 X \cong \Sigma\Omega X$ and the last equals X, since X is *torsion-free. □

8. Contractions

Definition 8.1. We say that a finite sequence of morphisms in $\mathscr{K}(R)$;

(8.1)
$$0 \longrightarrow X_n \xrightarrow{q_n} F_{n-1} \xrightarrow{f_{n-1}} F_{n-2} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{f_1} F_0 \xrightarrow{p_0} X_0 \longrightarrow 0$$

is $\mathscr{K}(R)$**-exact** if there are triangles

$$X_{i+1} \xrightarrow{q_{i+1}} F_i \xrightarrow{p_i} X_i \xrightarrow{\omega_i} X_{i+1}[1]$$

and equalities

$$f_i = q_i\, p_i$$

for $0 \le i \le n-1$. The $\mathscr{K}(R)$-exact sequence (8.1) can be described in a single diagram as

$$\begin{array}{ccccccccccccccc} & & & & X_{n-1} & & X_{n-2} & & & & X_1 & & & & \\ & & & & \uparrow{\scriptstyle p_{n-1}} & \searrow{\scriptstyle q_{n-1}} & \uparrow{\scriptstyle p_{n-2}} & \searrow & & & \uparrow{\scriptstyle p_1} & \searrow{\scriptstyle q_1} & & & \\ 0 & \longrightarrow & X_n & \xrightarrow{q_n} & F_{n-1} & \xrightarrow{f_{n-1}} & F_{n-2} & \longrightarrow & \cdots & \longrightarrow & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{p_0} & X_0 \longrightarrow 0. \end{array}$$

We also call the $\mathscr{K}(R)$-exact sequence (8.1) is a **partial** $\mathrm{Add}(R)$**-resolution of** X_0 if $F_i \in \mathrm{Add}(R)$ for all $0 \le i < n$. By definition an $\mathrm{Add}(R)$**-resolution of** X_0 **of length** $n-1$ is a partial $\mathrm{Add}(R)$-resolution with $X_n = 0$.

Note that, in the paper [7, Notation 3.2], we call a $\mathscr{K}(R)$-exact sequence an $(n+1)$-angle in $\mathscr{K}(R)$. However in the present paper we are interested only in the 'exactness' of the sequence, and not in the length n. For this reason we use the term '$\mathscr{K}(R)$-exact sequence' instead of $(n+1)$-angle.

If we are given such a $\mathscr{K}(R)$-exact sequence (8.1), we have a natural morphism $\widetilde{\omega_n} : X_0 \to X_n[n]$ that is defined by the composition $\omega_{n-1}[n-1]\omega_{n-2}[n-2]\cdots\omega_1[1]\omega_0$ of the morphisms in the relevant triangles. Notice that the morphism $\widetilde{\omega_n} : X_0 \to X_n[n]$ is uniquely determined by (8.1), which we call the **connecting morphism** of the $\mathscr{K}(R)$-exact sequence (8.1).

Theorem and Definition 8.2. *Suppose we are given a partial* $\mathrm{Add}(R)$*-resolution;*

(8.2)
$$\begin{array}{ccccccccccccccc} & & & & X_{n-1} & & X_{n-2} & & & & X_1 & & & & \\ & & & & \uparrow{\scriptstyle p_{n-1}} & \searrow{\scriptstyle q_{n-1}} & \uparrow{\scriptstyle p_{n-2}} & \searrow & & & \uparrow{\scriptstyle p_1} & \searrow{\scriptstyle q_1} & & & \\ 0 & \longrightarrow & X_n & \xrightarrow{q_n} & F_{n-1} & \xrightarrow{f_{n-1}} & F_{n-2} & \longrightarrow & \cdots & \longrightarrow & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{p_0} & X_0 \longrightarrow 0. \end{array}$$

Furthermore we assume that each $F_i \in \mathrm{Add}(R)$ *contains no null complex as a direct summand for* $0 \le i \le n-1$*. Then there is a triangle of the form*

$$X_n[n-1] \xrightarrow{\psi_n} \widetilde{F} \xrightarrow{\varphi_n} X_0 \xrightarrow{\widetilde{\omega_n}} X_n[n], \tag{8.3}$$

where $\widetilde{\omega_n}$ *is the connecting morphism of the sequence (8.2) and the following conditions are satisfied:*

(1) *There is an equality as underlying graded R-modules*

$$\widetilde{F} = F_{n-1}[n-1] \oplus F_{n-2}[n-2] \oplus \cdots \oplus F_1[1] \oplus F_0.$$

(2) *Let $in_i : F_i[i] \to \widetilde{F}$ and $pr_i : \widetilde{F} \to F_i[i]$ be respectively a natural injection and a natural projection of graded R-modules according to the direct sum decomposition in* (1)*. Denoting by $d_{\widetilde{F}}$ the differential mapping of $\widetilde{F}$, we have equalities of graded R-module homomorphisms;*

$$\begin{cases} pr_j \, d_{\widetilde{F}} \, in_i = 0 & \textit{for} \ \ 0 \le i \le j \le n-1, \\ pr_{i-1} \, d_{\widetilde{F}} \, in_i = f_i[i] & \textit{for} \ \ 1 \le i \le n-1. \end{cases}$$

(3) *The natural inclusion $in_0 : F_0 \to \widetilde{F}$ and the natural projection $pr_{n-1} : \widetilde{F} \to F_{n-1}[n-1]$ yield the morphisms in $\mathscr{K}(R)$ which make the following diagrams in $\mathscr{K}(R)$ commutative;*

$$\text{(8.4)} \qquad \begin{array}{ccc} X_n[n-1] & \xrightarrow{\psi_n} & \widetilde{F} \\ & {}_{q_n[n-1]}\searrow & \downarrow{\scriptstyle pr_{n-1}} \\ & & F_{n-1}[n-1] \end{array} \qquad\qquad \begin{array}{ccc} F_0 & & \\ {\scriptstyle in_0}\downarrow & \searrow^{p_0} & \\ \widetilde{F} & \xrightarrow[\varphi_n]{} & X_0 \end{array}$$

As an object of $\mathscr{K}(R)$, such a complex $\widetilde{F}$ is unique up to isomorphism.

We call $\widetilde{F}$ the **contraction** of the partial $\mathrm{Add}(R)$-resolution (8.2). The triangle (8.3) is called the **contracted triangle** of (8.2).

Proof. We prove the theorem by the induction on n. If $n = 1$, then there is a triangle

$$X_1 \xrightarrow{q_1} F_0 \xrightarrow{p_0} X_0 \xrightarrow{\omega_0} X_1[1].$$

Hence it suffices to set $\widetilde{F} = F_0$, $\psi_1 = q_1$ and $\varphi_1 = p_0$ in this case, and then the conditions (1) - (3) are satisfied in an obvious sense.

Now we assume $n > 1$. Setting $\widetilde{F'}$ as the contraction of the partial $\mathrm{Add}(R)$-resolution

$$0 \longrightarrow X_{n-1} \xrightarrow{q_{n-1}} F_{n-2} \xrightarrow{f_{n-2}} F_{n-3} \xrightarrow{f_{n-3}} \cdots F_1 \xrightarrow{f_1} F_0 \xrightarrow{p_0} X_0 \longrightarrow 0,$$

we assume that the theorem holds for this partial resolution and $\widetilde{F'}$. Then we have the following octahedron diagram:

(8.5)
$$
\begin{array}{ccccccc}
 & & \widetilde{F'} & = = = = & \widetilde{F'} & & \\
 & & \downarrow{\scriptstyle \iota_{n-1}} & & \downarrow{\scriptstyle \varphi_{n-1}} & & \\
X_n[n-1] & \xrightarrow{\psi_n} & \widetilde{F} & \xrightarrow{\varphi_n} & X_0 & \xrightarrow{\widetilde{\omega_n}} & X_n[n] \\
\| & & \downarrow{\scriptstyle pr_{n-1}} & & \downarrow{\scriptstyle \widetilde{\omega_{n-1}}} & & \| \\
X_n[n-1] & \xrightarrow{q_n[n-1]} & F_{n-1}[n-1] & \xrightarrow{p_{n-1}[n-1]} & X_{n-1}[n-1] & \xrightarrow{\omega_{n-1}[n-1]} & X_n[n] \\
 & & \downarrow{\scriptstyle \alpha_{n-1}[1]} & & \downarrow{\scriptstyle \psi_{n-1}[1]} & & \\
 & & \widetilde{F'}[1] & = = = = & \widetilde{F'}[1] & &
\end{array}
$$

where $\alpha_{n-1} = \psi_{n-1}\, p_{n-1}[n-2]$. In fact, the third column and the third row of this diagram are triangles by the induction hypothesis and the $\mathscr{K}(R)$-exactness assumption. The second column is a triangle given by setting $\widetilde{F} = Cone(\alpha_{n-1})$. The second row gives the desired triangle for the case of n. Since $\widetilde{F}$ is the mapping cone of the morphism α_{n-1}, it equals $F_{n-1}[n-1] \oplus \widetilde{F'}$ as an underlying graded R-module, and since $d_{F_{n-1}} = 0$, the summand $F_{n-1}[n-1]$ is mapped by $\alpha_{n-1}[1]$ into $\widetilde{F'}[1]$ under the differential $d_{\widetilde{F}}$. This proves $pr_{n-1}\, d_{\widetilde{F}}\, in_{n-1} = 0$ and the restriction of $d_{\widetilde{F}}$ on $\widetilde{F'}$ is its own differential $d_{\widetilde{F'}}$. It then follows from the induction hypothesis on $\widetilde{F'}$ that $pr_j\, d_{\widetilde{F}}\, in_i = 0$ holds for $0 \leq i \leq j \leq n-1$. If $1 \leq i \leq n-2$ then the equality $pr_{i-1}\, d_{\widetilde{F}}\, in_i = pr_{i-1}\, d_{\widetilde{F'}}\, in_i = f_i[i]$ holds by the induction hypothesis. To prove that $pr_{n-2}\, d_{\widetilde{F}}\, in_{n-1} = f_{n-1}[n-1]$, we note that $d_{\widetilde{F}}\, in_{n-1} = \alpha_{n-1}[1]$ and that $pr'_{n-2} : \widetilde{F'} \to F_{n-2}[n-2]$ is a chain map by the induction hypothesis. We consider the following diagram:

(8.6)
$$
\begin{array}{ccc}
F_{n-1}[n-1] & \xrightarrow{\alpha_{n-1}[1]} & \widetilde{F'}[1] \\
{\scriptstyle p_{n-1}[n-1]}\downarrow & \nearrow{\scriptstyle \psi_{n-1}[1]} & \downarrow{\scriptstyle pr'_{n-2}[1]} \\
X_{n-1}[n-1] & \xrightarrow[q_{n-1}[n-1]]{} & F_{n-2}[n-1]
\end{array}
$$

The upper left triangle is commutative by the definition of α_{n-1}, and the lower right triangle is also commutative by the induction hypothesis for $\widetilde{F'}$. Therefore the square above is a commutative diagram, and thus we have $pr'_{n-2}[1]\, \alpha_{n-1}[1] = q_{n-1}[n-1]\, p_{n-1}[n-1] = f_{n-1}[n-1]$. This proves $pr_{n-2}\, d_{\widetilde{F}}\, in_{n-1} = f_{n-1}[n-1]$, and the conditions (1) and (2) are proved.

We can see from (2) that $d_{\widetilde{F}}\, in_0 = 0$ and $pr_{n-1}\, d_{\widetilde{F}} = 0$, hence in_0 and pr_{n-1} are chain maps. The commutativity of the left triangle in (8.4) follows from the diagram (8.5). To prove that the right triangle in (8.4) is also commutative, note that $p_0 = \varphi_{n-1}\, in'_0$

holds by the induction hypothesis, where $in'_0 : F_0 \to \widetilde{F'}$ is the natural injection. Note form the diagram (8.5) that $\varphi_n \,\iota_{n-1} = \varphi_{n-1}$ and $\iota_{n-1}\, in'_0 = in_0$. Thus we obtain $\varphi_n \, in_0 = \varphi_n \,\iota_{n-1}\, in'_0 = \varphi_{n-1}\, in'_0 = p_0$.

Since the connecting morphism $\widetilde{\omega_n}$ is uniquely determined by the $\mathscr{K}(R)$-exact sequence, noting that $\widetilde{F} \cong Cone(\widetilde{\omega_n}[-1])$ in $\mathscr{K}(R)$, we see that $\widetilde{F}$ is unique up to isomorphism in $\mathscr{K}(R)$. □

Remark 8.3. In Theorem 8.2 we have shown that the contraction $\widetilde{F}$ is unique up to isomorphism. However, as to ψ_n and φ_n in the triangle (8.3), they are not necessarily unique as a morphism in $\mathscr{K}(R)$.

Another interpretation of the conditions (1)-(3) in the theorem is the following: Now returning to the setting of Theorem 8.2, the contraction of the $\mathscr{K}(R)$-exact sequence (8.2) is $F_{n-1}[n-1] \oplus \cdots \oplus F_1[1] \oplus F_0$ as an underlying graded R-module and the differential $d_{\widetilde{F}}$ is given by a matrix of the form;

$$d_{\widetilde{F}} = \begin{pmatrix} 0 & 0 & \cdots\cdots & 0 & 0 \\ f_{n-1}[n-1] & 0 & \cdots\cdots & 0 & 0 \\ a_{n-1\,n-3} & f_{n-2}[n-2] & \cdots\cdots & 0 & 0 \\ a_{n-1\,n-4} & a_{n-2\,n-4} & \cdots\cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1\,0} & a_{n-2\,0} & \cdots\cdots & f_1[1] & 0 \end{pmatrix}, \tag{8.7}$$

where each $a_{i\,j} : F_i[i] \to F_j[j+1]$ is a graded R-homomorphism. On the other hand, the commutativity of the diagrams (8.4) says that as underlying graded R-module homomorphisms ψ_n and φ_n are represented respectively by the following matrices:

$$\psi_n = \begin{pmatrix} q_n[n-1] \\ a_{n\,n-2} \\ \vdots \\ a_{n\,0} \end{pmatrix} : X_n[n-1] \longrightarrow F_{n-1}[n-1] \oplus F_{n-2}[n-2] \oplus \cdots \oplus F_1[1] \oplus F_0$$

$$\varphi_n = \begin{pmatrix} b_{n-1\,0} & \cdots & b_{1\,0} & p_0 \end{pmatrix} : F_{n-1}[n-1] \oplus F_{n-2}[n-2] \oplus \cdots \oplus F_1[1] \oplus F_0 \longrightarrow X_0$$

for some graded R-homomorphisms $a_{n\,i} : X_n[n-1] \to F_i[i]$ and $b_{i\,0} : F_i[i] \to X_0$.

Definition 8.4. Assume that we have a commutative diagram

$$\begin{array}{ccccccccccccccc} 0 & \longrightarrow & X_n & \xrightarrow{q_n^F} & F_{n-1} & \xrightarrow{f_{n-1}} & \cdots & \longrightarrow & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{p_0^F} & X_0 & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle t_n} & & \uparrow{\scriptstyle s_{n-1}} & & & & \uparrow{\scriptstyle s_1} & & \uparrow{\scriptstyle s_0} & & \uparrow{\scriptstyle t_0} & & \\ 0 & \longrightarrow & Y_n & \xrightarrow{q_n^G} & G_{n-1} & \xrightarrow{g_{n-1}} & \cdots & \longrightarrow & G_1 & \xrightarrow{g_1} & G_0 & \xrightarrow{p_0^G} & Y_0 & \longrightarrow & 0, \end{array} \tag{8.8}$$

where the rows are partial Add(R)-resolutions. We say that the diagram (8.8) gives a morphism between the partial Add(R)-resolutions if there are commutative diagrams

$$\begin{array}{ccccccc} X_{i+1} & \xrightarrow{q^F_{i+1}} & F_i & \xrightarrow{p^F_i} & X_i & \xrightarrow{\omega^F_i} & X_{i+1}[1] \\ \uparrow{\scriptstyle t_{i+1}} & & \uparrow{\scriptstyle s_i} & & \uparrow{\scriptstyle t_i} & & \uparrow{\scriptstyle t_{i+1}} \\ Y_{i+1} & \xrightarrow{q^G_{i+1}} & G_i & \xrightarrow{p^G_i} & Y_i & \xrightarrow{\omega^G_i} & Y_{i+1}[1], \end{array}$$

where each row is a triangle in $\mathscr{K}(R)$ for $0 \leq i < n$, and $f_i = q^F_i p^F_i$, $g_i = q^G_i p^G_i$ for $1 \leq i < n$.

In such a case we have a morphism $\widetilde{s}$ between the contractions with the diagram;

$$\begin{array}{ccccccc} X_n[n-1] & \xrightarrow{\psi^F_n} & \widetilde{F} & \xrightarrow{\varphi^F_n} & X_0 & \xrightarrow{\widetilde{\omega^F_n}} & X_n[n] \\ \uparrow{\scriptstyle t_n[n-1]} & & \uparrow{\scriptstyle \widetilde{s}} & & \uparrow{\scriptstyle t_0} & & \uparrow{\scriptstyle t_n[n]} \\ Y_n[n-1] & \xrightarrow{\psi^G_n} & \widetilde{G} & \xrightarrow{\varphi^G_n} & Y_0 & \xrightarrow{\widetilde{\omega^G_n}} & Y_n[n]. \end{array} \tag{8.9}$$

Since the morphisms t_0 and t_n are given beforehand, such a morphism $\widetilde{s}$ obviously exists so that the diagram (8.9) will be commutative. Unfortunately note that it is not unique in general.

But we can prove the following theorem.

Theorem 8.5. *Under the circumstances in Definition 8.4, we furthermore assume that all the $F_i, G_i (1 \leq i < n)$ do not contain any null complexes as direct summands. Then we can take a morphism $\widetilde{s} : \widetilde{G} \to \widetilde{F}$ so that it is represented by the following type of lower triangle matrix as an underlying graded R-module homomorphism according to the direct sum decompositions $\widetilde{G} = G_{n-1}[n-1] \oplus G_{n-2}[n-2] \oplus G_0$ and $\widetilde{F} = F_{n-1}[n-1] \oplus F_{n-2}[n-2] \oplus F_0$:*

$$\begin{pmatrix} s_{n-1}[n-1] & 0 & 0 & \ldots\ldots & 0 \\ * & s_{n-2}[n-2] & 0 & \ldots\ldots & 0 \\ * & * & s_{n-3}[n-3] & \ldots\ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \ldots\ldots & s_0 \end{pmatrix}$$

In this case the following diagram is commutative:

$$\begin{array}{ccc} \widetilde{F} & \xrightarrow{pr^F_{n-1}} & F_{n-1}[n-1] \\ \uparrow{\scriptstyle \widetilde{s}} & & \uparrow{\scriptstyle s_{n-1}[n-1]} \\ \widetilde{G} & \xrightarrow{pr^G_{n-1}} & G_{n-1}[n-1] \end{array}$$

Proof. The last part of the theorem follows from the first, since pr^F_{n-1} and pr^G_{n-1} are represented by matrices of the form $(1\ 0\ \cdots\ 0)$.

Under the same notation as in the proof of Theorem 8.2, $\widetilde{F}$ and $\widetilde{G}$ are the mapping cones of the morphisms $\alpha^F_{n-1} = \psi^F_{n-1} p^F_{n-1}[n-2]$ and $\alpha^G_{n-1} = \psi^G_{n-1} p^G_{n-1}[n-2]$ respectively. By the induction hypothesis for $n-1$ we may assume that we have such an $\widetilde{s'}$ that is represented by a lower triangle matrix and that makes the following diagram commutative.

$$
\begin{array}{ccccccc}
X_{n-1}[n-2] & \xrightarrow{\psi^F_{n-1}} & \widetilde{F'} & \xrightarrow{\varphi^F_{n-1}} & X_0 & \xrightarrow{\widetilde{\omega^F_{n-1}}} & X_{n-1}[n] \\
{\scriptstyle t_{n-1}[n-2]}\uparrow & & \uparrow{\scriptstyle \widetilde{s'}} & & \uparrow{\scriptstyle t_0} & & \uparrow{\scriptstyle t_{n-1}[n-1]} \\
Y_{n-1}[n-2] & \xrightarrow{\psi^G_{n-1}} & \widetilde{G'} & \xrightarrow{\varphi^G_{n-1}} & Y_0 & \xrightarrow{\widetilde{\omega^G_{n-1}}} & Y_{n-1}[n-1].
\end{array}
$$

Since there is a triangle $X_n \longrightarrow F_{n-1} \xrightarrow{p^F_{n-1}} X_{n-1} \longrightarrow X_n[1]$ in $\mathscr{K}(R)$, we see that X_n is isomorphic to the mapping cone of $p^F_{n-1}[-1]$ i.e. $F_{n-1} \oplus X_{n-1}[-1]$ is its underlying graded R-module and it has the differential $d_{X_n} = \begin{pmatrix} 0 & 0 \\ p^F_{n-1} & d_{X_{n-1}[-1]} \end{pmatrix}$. This is similar to Y_n, hence $Y_n \cong G_{n-1} \oplus Y_{n-1}[-1]$ as an underlying graded R-module with $d_{Y_n} = \begin{pmatrix} 0 & 0 \\ p^G_{n-1} & d_{Y_{n-1}[-1]} \end{pmatrix}$. Note that $X_{n-1}[-1]$ is a subcomplex of this mapping cone, and F_{n-1} is a quotient of it. Since t_n maps the subcomplex $X_{n-1}[-1]$ into the subcomplex $Y_{n-1}[n-1]$, we can see that t_n is represented by a matrix

$$
G_{n-1} \oplus Y_{n-1}[-1] \xrightarrow{\begin{pmatrix} s_{n-1} & 0 \\ u & t_{n-1}[-1] \end{pmatrix}} F_{n-1} \oplus X_{n-1}[-1],
$$

where $u : G_{n-1} \longrightarrow X_{n-1}[-1]$ is a graded R-homomorphism. Identifying those relevant complexes under such isomorphisms, we also see from the inductive construction of $\widetilde{F}$ in the proof of Theorem 8.2 that the morphism $\psi^F_n : X_n[n-1] \longrightarrow \widetilde{F}$ is given by the chain map

$$
F_{n-1}[n-1] \oplus X_{n-1}[n-2] \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & \psi^F_{n-1} \end{pmatrix}} F_{n-1}[n-1] \oplus \widetilde{F'}.
$$

Similarly $\varphi^F_n : \widetilde{F} \longrightarrow X_0$ is represented by

$$
F_{n-1}[n-1] \oplus \widetilde{F'} \xrightarrow{\begin{pmatrix} 0 & \varphi^F_{n-1} \end{pmatrix}} X_0.
$$

Finally it is easy to see that the following diagram is commutative:

$$\begin{array}{ccccc}
F_{n-1}[n-1] \oplus X_{n-1}[n-2] & \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & \psi^F_{n-1} \end{pmatrix}} & F_{n-1}[n-1] \oplus \widetilde{F'} & \xrightarrow{\begin{pmatrix} 0 & \varphi^F_{n-1} \end{pmatrix}} & X_0 \\
{\scriptstyle \begin{pmatrix} s_{n-1}[n-1] & 0 \\ u[n-1] & t_{n-1}[n-2] \end{pmatrix}} \Big\uparrow & & {\scriptstyle \begin{pmatrix} s_{n-1}[n-1] & 0 \\ \psi^F_{n-1}u[n-1] & \widetilde{s'} \end{pmatrix}} \Big\uparrow & & \Big\uparrow {\scriptstyle t_0} \\
G_{n-1}[n-1] \oplus Y_{n-1}[n-2] & \xrightarrow[\begin{pmatrix} 1 & 0 \\ 0 & \psi^G_{n-1} \end{pmatrix}]{} & G_{n-1}[n-1] \oplus \widetilde{G'} & \xrightarrow[\begin{pmatrix} 0 & \varphi^G_{n-1} \end{pmatrix}]{} & Y_0
\end{array}$$

Therefore we can take the matrix $\begin{pmatrix} s_{n-1}[n-1] & 0 \\ \psi^F_{n-1}u[n-1] & \widetilde{s'} \end{pmatrix}$ as $\widetilde{s}$. Since $\widetilde{s'}$ is taken to be a lower triangle matrix by the induction hypothesis, so is $\widetilde{s}$. $\square$

9. Remarks on partial Add(R)-resolutions

Definition 9.1.
(1) We say that a partial Add(R)-resolution (8.2) is **split** if each q_i in Definition 8.1 has a left inverse, i.e. q_i is a split monomorphism, for all $1 \leq i \leq n$. This is equivalent to $\omega_i = 0$ for all $0 \leq i < n$, with the notation in Definition 8.1.

(2) We say that a partial Add(R)-resolution (8.2) is **degenerate** if one can choose the differential $d_{\widetilde{F}}$ as it satisfies $pr_j \cdot d_{\widetilde{F}} \cdot in_i = 0$ unless $j = i-1$ for all $1 \leq i \leq n-1$ under the notation of Theorem 8.2. This is equivalent to saying that one can take the differential of the form

$$d_{\widetilde{F}} = \begin{pmatrix} 0 & 0 & 0 & \cdots\cdots & 0 & 0 \\ f_{n-1}[n-1] & 0 & 0 & \cdots\cdots & 0 & 0 \\ 0 & f_{n-2}[n-2] & 0 & \cdots\cdots & 0 & 0 \\ 0 & 0 & f_{n-3}[n-3] & \cdots\cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots\cdots & f_1[1] & 0 \end{pmatrix} \tag{9.1}$$

as a grade R-module mapping from $\widetilde{F} = F_{n-1}[n-1] \oplus F_{n-2}[n-2] \oplus \cdots \oplus F_1[1] \oplus F_0$ to $\widetilde{F}[1]$. Note in this case that we have an equality

$$\widetilde{F} = \coprod_{i \in \mathbb{Z}} \left[0 \longrightarrow F^i_{n-1} \xrightarrow{f^i_{n-1}} F^i_{n-2} \longrightarrow \cdots \longrightarrow F^i_1 \xrightarrow{f^i_1} F^i_0 \longrightarrow 0 \right] [-i].$$

The following proposition will be necessary in the later argument of this paper.

Proposition 9.2. *Let*

$$0 \longrightarrow F_{n-1} \xrightarrow{f_{n-1}} F_{n-2} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{f_1} F_0 \xrightarrow{p_0} X_0 \longrightarrow 0 \tag{9.2}$$

be an $\mathrm{Add}(R)$*-resolution of length* $n-1$ *and let* $\widetilde{F}$ *be its contraction. Assume that* $n \geq 2$ *and* f_{n-1} *has a left inverse in* $\mathscr{K}(R)$*. Then* $X_0 \cong \widetilde{F}$ *and the morphism* $pr_{n-1} : \widetilde{F} \longrightarrow F_{n-1}[n-1]$ *is zero in* $\mathscr{K}(R)$*.*

Proof. The isomorphism $X_0 \cong \widetilde{F}$ follows from the contraction sequence (8.3) in Theorem 8.2 by setting $X_n = 0$. Note that pr_{n-1} is represented by the matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} : F_{n-1}[n-1] \oplus \cdots \oplus F_1[1] \oplus F_0 \longrightarrow F_{n-1}[n-1].$$

Let v be a left inverse of f_{n-1}, i.e. $v : F_{n-2} \to F_{n-1}$ such that $vf_{n-1} = 1_{F_{n-1}}$ and set $\widetilde{v} : \widetilde{F}[1] \to F_{n-1}[n-1]$ as a graded R-homomorphism given by the matrix

$$\begin{pmatrix} 0 & v[n-1] & 0 & \cdots & 0 \end{pmatrix} : F_{n-1}[n] \oplus F_{n-2}[n-1] \oplus \cdots \oplus F_0 \longrightarrow F_{n-1}[n-1].$$

Then, since the differential $d_{\widetilde{F}}$ is represented by the matrix (8.7), it is easy to see that $pr_{n-1} = \widetilde{v} d_{\widetilde{F}}$. Hence pr_{n-1} is null homotopic. □

Corollary 9.3. *Assume that the* $\mathrm{Add}(R)$*-resolution (9.2) is split where* $n \geq 2$*. Then* $pr_{n-1} = 0$ *in* $\mathscr{K}(R)$*.*

Lemma 9.4. *We assume that the following conditions are satisfied for the partial* $\mathrm{Add}(R)$*-resolution (8.2) :*

(1) X_0, X_n *belong to* $\mathrm{Add}(R)$ *and all the* $F_i \in \mathrm{Add}(R)$*,* X_0 *and* X_n *have no null complexes as direct summands for* $0 \leq i < n$*.*

(2) *As a sequence of graded* R*-modules, the sequence*

$$0 \longrightarrow X_n \xrightarrow{q_n} F_{n-1} \xrightarrow{f_{n-1}} F_{n-2} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{f_1} F_0 \xrightarrow{p_0} X_0 \longrightarrow 0$$

is exact.

Then the partial $\mathrm{Add}(R)$*-resolution is degenerate. The contracted triangle (8.3) is realized by the morphisms represented by the following form of underlying graded* R*-module homomorphisms:*

$$\psi_n = \begin{pmatrix} q_n[n-1] \\ 0 \\ \vdots \\ 0 \end{pmatrix} : X_n[n-1] \longrightarrow \widetilde{F} = F_{n-1}[n-1] \oplus F_{n-2}[n-2] \oplus \cdots \oplus F_1[1] \oplus F_0$$

$$\varphi_n = \begin{pmatrix} 0 & \cdots & 0 & p_0 \end{pmatrix} : \widetilde{F} = F_{n-1}[n-1] \oplus F_{n-2}[n-2] \oplus \cdots \oplus F_1[1] \oplus F_0 \longrightarrow X_0$$

Proof. Set $d_{\widetilde{F}}$ as in (9.1) and we see by a straightforward computation that $d_{\widetilde{F}}^2 = 0$, $d_{\widetilde{F}} \psi_n = 0$ and $\varphi_n d_{\widetilde{F}} = 0$ where ψ_n and φ_n are given as in the lemma. (Note in this computation we need $f_{n-1} q_n = 0$ and $p_0 f_1 = 0$ as graded R-module homomorphisms. In general we see that they are zero in $\mathscr{K}(R)$, that is, null homotopic. But it does not mean they are zero homomorphisms. The assumption that all $X_i, F_j \in \mathrm{Add}(R)$ have no null summands is necessary to conclude they are zero homomorphisms.) Therefore the matrices given in the lemma define chain homomorphisms. It is then easy to see that the sequence

$$X_n[n-1] \xrightarrow{\psi_n} \widetilde{F} \xrightarrow{\varphi_n} X_0 \xrightarrow{0} X_n[n]$$

is a triangle in $\mathscr{K}(R)$. $\square$

Corollary 9.5. *If a partial* $\mathrm{Add}(R)$*-resolution is split, then it is degenerate.*

We should note that all partial $\mathrm{Add}(R)$-resolutions of length $n \leq 2$ are degenerate. In fact, if $n = 2$ then the $\mathscr{K}(R)$-exact sequence is $0 \longrightarrow X_2 \longrightarrow F_1 \xrightarrow{f_1} F_0 \longrightarrow X \longrightarrow 0$, and $\widetilde{F}$ is the mapping cone of f_1, therefore the sequence is degenerate.

Note also that, even if there is a $\mathscr{K}(R)$-exact sequence

$$0 \longrightarrow F_n \xrightarrow{f_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{f_1} F_0 \longrightarrow X \longrightarrow 0$$

for assigned $F_i \in \mathrm{Add}(R)$ and f_i, the rightmost complex X is not uniquely determined. In fact, X depends not only on f_i but also on p_i, q_i with $f_i = p_i q_i$ as in Definition 8.2.

For $X \in \mathscr{K}(R)$ and for an integer $n > 0$, we define the n-th syzygy and cosyzygy by the induction on n;

$$\Omega^0 X = \Sigma^0 X = X, \quad \Omega^n X = \Omega(\Omega^{n-1} X), \quad \Sigma^n X = \Sigma(\Sigma^{n-1} X).$$

Recall from Definitions 7.3 and 7.8 that $\Omega^n X$ and $\Sigma^n X$ are uniquely determined as objects in $\underline{\mathscr{K}(R)}$, or in other words they are unique up to $\mathrm{Add}(R)$-summands as objects in $\mathscr{K}(R)$. Actually they define the functors $\Omega^n, \Sigma^n : \underline{\mathscr{K}(R)} \to \underline{\mathscr{K}(R)}$, and Theorem 7.11 assures that (Σ^n, Ω^n) is an adjoint pair for each $n > 0$.

Definition 9.6. Let $X \in \mathscr{K}(R)$ and take a right $\mathrm{Add}(R)$-approximation $p_0 : F_0 \to X$. We embed p_0 into a triangle to get the first syzygy $\Omega^1 X$;

$$\Omega^1 X \xrightarrow{q_1} F_0 \xrightarrow{p_0} X \xrightarrow{\omega_1^X} \Omega^1 X[1].$$

Similarly but as for the dual version to this, we have a triangle for any $Y \in \mathscr{K}(R)$;

$$\Sigma Y[-1] \xrightarrow{\omega_{-1}^Y} Y \xrightarrow{q_{-0}} G_0 \xrightarrow{p_{-0}} \Sigma Y,$$

where q_{-0} is a left $\mathrm{Add}(R)$-approximation. In such a way we have morphisms ω_1^X and ω_{-1}^Y. Now let n be a positive integer. We define inductively

$$\omega_n^X = \omega_{n-1}^{\Sigma X[1]} \omega_1^X \ : \ X \longrightarrow \Omega^n X[n], \quad \omega_{-n}^Y = \omega_{-1}^Y \omega_{-n+1}^{\Sigma Y[-1]} \ : \ \Sigma^n Y[-n] \longrightarrow Y.$$

Let X be an arbitrary object in $\mathscr{K}(R)$. Note from the definition of Ω^i that there are triangles

$$\Omega^{i+1} X \xrightarrow{q_{i+1}} F_i \xrightarrow{p_i} \Omega^i X \xrightarrow{\omega_1^{\Omega^i X}} \Omega^{i+1} X[1],$$

where $F_i \in \mathrm{Add}(R)$ and p_i is a right $\mathrm{Add}(R)$-approximation of $\Omega^i X$ for all $i \geq 0$. Hence, when n is a positive integer, we have a partial $\mathrm{Add}(R)$-resolution of the form;

$$0 \longrightarrow \Omega^n X \xrightarrow{q_n} F_{n-1} \xrightarrow{f_{n-1}} \cdots \longrightarrow F_1 \xrightarrow{f_1} F_0 \xrightarrow{p_0} X \longrightarrow 0, \tag{9.3}$$

where $F_i \in \mathrm{Add}(R)$ and $f_i = q_i p_i$ for $0 \leq i < n$. We note here that we may assume that all the F_i $(0 \leq i < n)$ have zero differentials, because we can take them up to isomorphisms in $\underline{\mathscr{K}(R)}$. (Cf. Theorem 5.8.) Hence we may assume that $H(F_i) = F_i$ for all $0 \leq i < n$. Note also that ω_n^X defined in Definition 9.6 is the connecting morphism of the partial $\mathrm{Add}(R)$-resolution (9.3).

The following theorem is a restatement of Theorem 8.2.

Theorem 9.7. *Under the circumstances above, there is a triangle in $\mathscr{K}(R)$;*

$$\Omega^n X[n-1] \xrightarrow{\psi_n} \widetilde{F} \xrightarrow{\varphi_n} X \xrightarrow{\omega_n^X} \Omega^n X[n],$$

where the morphisms $\psi_n : \Omega^n X[n-1] \to \widetilde{F}$ and $\varphi_n : \widetilde{F} \to X$ make the following diagrams commutative:

$$\begin{array}{ccc} \Omega^n X[n-1] & \xrightarrow{\psi_n} & \widetilde{F} \\ & \searrow^{q_n[n-1]} & \downarrow{\scriptstyle pr_{n-1}} \\ & & F_{n-1}[n-1], \end{array} \qquad \begin{array}{ccc} F_0 & & \\ \downarrow{\scriptstyle in_0} & \searrow^{p_0} & \\ \widetilde{F} & \xrightarrow{\varphi_n} & X. \end{array} \tag{9.4}$$

We shall make several remarks on the partial $\mathrm{Add}(R)$-resolution (9.3). Firstly we see from Lemma 7.2 and from the above observation that the following is an exact sequence of graded R-modules;

$$0 \longrightarrow H(\Omega^n X) \xrightarrow{H(q_n)} F_{n-1} \xrightarrow{f_{n-1}} \cdots \longrightarrow F_1 \xrightarrow{f_1} F_0 \xrightarrow{H(p_0)} H(X) \longrightarrow 0,$$

which means that there are exact sequences of R-modules

$$0 \longrightarrow H^i(\Omega^n X) \xrightarrow{H^i(q_n)} F^i_{n-1} \xrightarrow{f^i_{n-1}} \cdots \longrightarrow F^i_1 \xrightarrow{f^i_1} F^i_0 \xrightarrow{H^i(p_0)} H^i(X) \longrightarrow 0,$$

for all $i \in \mathbb{Z}$. The diagram (9.4) induces the commutative diagram of cohomology modules:

$$\begin{array}{ccc} H(\Omega^n X)[n-1] & \xrightarrow{H(\psi_n)} & H(\widetilde{F}) \\ & \searrow^{H(q_n)[n-1]} & \downarrow{\scriptstyle H(pr_{n-1})} \\ & & F_{n-1}[n-1], \end{array} \qquad \begin{array}{ccc} F_0 & & \\ \downarrow{\scriptstyle H(in_0)} & \searrow^{H(p_0)} & \\ H(\widetilde{F}) & \xrightarrow{H(\varphi_n)} & H(X). \end{array}$$

Since $H(q_n)$ is injective, so is $H(\psi_n)$. Similarly $H(\varphi_n)$ is surjective, as $H(p_0)$ is surjective. As a consequence, it follows from the contracted triangle in Theorem 9.7 that there is an exact sequence of graded R-modules;

$$0 \longrightarrow H(\Omega^n X)[n-1] \xrightarrow{H(\psi_n)} H(\widetilde{F}) \xrightarrow{H(\varphi_n)} H(X) \longrightarrow 0\ .$$

Example 9.8. Let M be a finitely generated R-module and

(9.5) $$X = \left[\ \cdots \longrightarrow P_n \xrightarrow{u_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{u_1} P_0 \longrightarrow 0\ \right]$$

be an R-projective resolution of M, i.e. $X \in \mathscr{K}(R)$ and there is a quasi-isomorphism $X \to M$. In this case it is obvious to see that $\Omega^n X$ is the truncated complex $\left[\ \cdots \longrightarrow P_{n+1} \xrightarrow{u_n} P_n \longrightarrow 0\ \right]$, and there is a $\mathscr{K}(R)$-exact sequence

$$0 \longrightarrow \Omega^n X \longrightarrow P_{n-1} \xrightarrow{u_{n-1}} P_{n-2} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{u_1} P_0 \longrightarrow X \longrightarrow 0.$$

In this case the contraction of this partial $\mathrm{Add}(R)$-resolution is the complex

$$0 \longrightarrow P_{n-1} \longrightarrow P_{n-2} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0,$$

hence it is degenerate.

Even for such natural constructions we should remark that there are partial $\mathrm{Add}(R)$-resolutions of the form (9.3) that are not degenerate.

Example 9.9. Let M be a finitely generated R-module and X a projective resolution of M given as in (9.5). We consider a complex of length one;

$$Y = \left[\ 0 \longrightarrow P_1 \xrightarrow{u_1} P_0 \longrightarrow 0\ \right].$$

As we remarked in Example 7.12 we see that

$$\Omega^n Y = \left[\ 0 \longrightarrow P_{n+1} \xrightarrow{u_{n+1}} P_n \longrightarrow 0\ \right].$$

In fact, set

$$f_{i+1} = \begin{pmatrix} u_{i+1} & 0 \\ 0 & u_{i+3}[1] \end{pmatrix} \ : F_{i+1} := P_{i+1} \oplus P_{i+3}[1] \longrightarrow F_i := P_i \oplus P_{i+2}[1]$$

for $i \geq 0$, where each F_i is a complex with zero differential mappings. Furthermore we set

$$p_0 = \begin{pmatrix} 1 & 0 \\ 0 & u_2[1] \end{pmatrix} \ : F_0 = P_0 \oplus P_2[1] \longrightarrow Y = P_0 \oplus P_1[1],$$

and

$$q_n = \begin{pmatrix} u_n & 0 \\ 0 & 1 \end{pmatrix} \ : \Omega^n Y = P_n \oplus P_{n+1}[1] \longrightarrow F_{n-1} = P_{n-1} \oplus P_{n+1}[1].$$

Then we have a partial $\mathrm{Add}(R)$-resolution

$$0 \longrightarrow \Omega^n Y \xrightarrow{q_n} F_{n-1} \xrightarrow{f_{n-1}} \cdots \longrightarrow F_1 \xrightarrow{f_1} F_0 \xrightarrow{p_0} Y \longrightarrow 0,$$

as in (9.3).

In this example we can observe that if $n \geq 3$ then the partial $\mathrm{Add}(R)$-resolution is never degenerated.

For example, in the case $n = 3$, setting a graded R-module homomorphism

$$g = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} : F_2[2] = P_2[2] \oplus P_4[3] \longrightarrow F_0[1] = P_0[1] \oplus P_2[2],$$

we can see that the differential of $\widetilde{F}$ is given by

$$d_{\widetilde{F}} = \begin{pmatrix} 0 & 0 & 0 \\ f_2[2] & 0 & 0 \\ g & f_1[1] & 0 \end{pmatrix} : \widetilde{F} = F_2[2] \oplus F_1[1] \oplus F_0 \longrightarrow \widetilde{F}[1] = F_2[3] \oplus F_1[2] \oplus F_0[1],$$

which shows that the sequence is not degenerate.

Definition 9.10. We say that a partial $\mathrm{Add}(R)$-resolution

$$\begin{array}{ccccccccccccccc}
 & & & & X_{n-1} & & X_{n-2} & & & & X_1 & & & & \\
 & & & & \uparrow{\scriptstyle p_{n-1}} & \overset{q_{n-1}}{\searrow} & \uparrow{\scriptstyle p_{n-2}} & \searrow & & & \uparrow{\scriptstyle p_1} & \overset{q_1}{\searrow} & & & \\
0 & \longrightarrow & X_n & \xrightarrow{q_n} & F_{n-1} & \xrightarrow{f_{n-1}} & F_{n-2} & \longrightarrow & \cdots & \longrightarrow & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{p_0} & X_0 \longrightarrow 0.
\end{array}$$

is **generically split** if the localized $\mathscr{K}(R)$-exact sequence

$$\begin{array}{ccccccccccccc}
 & & & & S^{-1}X_{n-1} & & S^{-1}X_{n-2} & & S^{-1}X_1 & & & & \\
 & & & & {\scriptstyle S^{-1}p_{n-1}}\uparrow & \overset{S^{-1}q_{n-1}}{\searrow} & \uparrow & \searrow & & \overset{S^{-1}q_1}{\searrow} & & & \\
0 & \longrightarrow & S^{-1}X_n & \xrightarrow[S^{-1}q_n]{} & S^{-1}F_{n-1} & \xrightarrow[S^{-1}f_{n-1}]{} & S^{-1}F_{n-2} & \longrightarrow & \cdots & \xrightarrow[S^{-1}f_1]{} & S^{-1}F_0 & \xrightarrow[S^{-1}p_0]{} & S^{-1}X_0 \longrightarrow 0
\end{array}$$

is split in $\mathscr{K}(S^{-1}R)$ in the sense of Definition 9.1(1), where $S = R \setminus \bigcup_{\mathfrak{p} \in \mathrm{Ass}(R)} \mathfrak{p}$.

Example 9.11. Let a be an element of R. Then for the complex $X_1 := [\ 0 \to R \xrightarrow{a} R \to 0\ ]$ there are triangles;

$$R \xrightarrow{a} R \longrightarrow X_1 \longrightarrow R[1], \quad X_1 \longrightarrow R[1] \xrightarrow{a} R[1] \longrightarrow X_1[1].$$

Hence $X_0 := R[1]$ has the following type of finite $\mathrm{Add}(R)$-resolution:

$$0 \longrightarrow R \xrightarrow{a} R \xrightarrow{0} R[1] \xrightarrow{a} X_0 = R[1] \longrightarrow 0.$$

If a is a non-zero divisor then this resolution is generically split. However whenever a is a non-unit, the sequence is not split and not degenerate.

Example 9.12. Let $a, b \in R$ and assume that $\{a, b\}$ is a regular sequence on R of length two. Now let

$$X_0 = [\ 0 \longrightarrow R^2 \xrightarrow{(a\ b)} R \longrightarrow 0\ ], \quad X_1 = [\ 0 \longrightarrow R \xrightarrow{\binom{b}{-a}} R^2 \longrightarrow 0\ ],$$

and note that X_0 is *torsion-free but not *reflexive, while X_1 is not *torsion-free. One can easily see that there is an $\mathrm{Add}(R)$-resolution of X_0;

$$\begin{array}{ccccccccccc} & & & & X_1 & & & & & & \\ & & & & \uparrow{\scriptstyle p_1} & \searrow & & & & & \\ 0 & \longrightarrow & R & \xrightarrow{\binom{b}{-a}} & R^2 & \xrightarrow[\begin{pmatrix}0&0\\a&b\end{pmatrix}]{} & R[1]\oplus R & \xrightarrow{p_0} & X_0 & \longrightarrow & 0, \end{array}$$

where p_0, p_1 are chain maps defined respectively as

$$\begin{array}{ccccccc} 0 & \longrightarrow & R & \xrightarrow{0} & R & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \binom{b}{-a}} & & \downarrow{\scriptstyle 1} & & \\ 0 & \longrightarrow & R^2 & \xrightarrow[(a\ b)]{} & R & \longrightarrow & 0, \end{array} \qquad \begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & R^2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle 1} & & \\ 0 & \longrightarrow & R & \xrightarrow[\binom{b}{-a}]{} & R^2 & \longrightarrow & 0. \end{array}$$

It is easy to see that p_0 and p_1 are right $\mathrm{Add}(R)$-approximations, hence $\Omega X_0 = X_1$ and $\Omega^2 X_0 = 0$ in $\underline{\mathscr{K}(R)}$. We should notice that the $\mathrm{Add}(R)$-resolution above is not a split sequence, but generically split.

10. Counit morphism for the adjoint pair (Σ^n, Ω^n)

It follows from Theorem 7.11 that there is an isomorphism

$$\mathrm{Hom}_{\underline{\mathscr{K}(R)}}(\Sigma^{n-i}\Omega^n X, \Omega^i X) \cong \mathrm{Hom}_{\underline{\mathscr{K}(R)}}(\Omega^n X, \Omega^n X), \tag{10.1}$$

for all $X \in \mathscr{K}(R)$ and $0 \le i \le n$. Thus we can take a morphism in $\mathscr{K}(R)$;

$$\pi_X^{(n,i)} : \Sigma^{n-i}\Omega^n X \to \Omega^i X$$

which yields a unique element of $\mathrm{Hom}_{\underline{\mathscr{K}(R)}}(\Sigma^{n-i}\Omega^n X, \Omega^i X)$ that corresponds to the identity on $\Omega^n X$ in the right hand side of (10.1).

If $i = 0$, then $\underline{\pi_X^{(n,0)}} \in \mathrm{Hom}_{\underline{\mathscr{K}(R)}}(\Sigma^n\Omega^n X, X)$ is a counit morphism for the adjoint pair (Σ^n, Ω^n). If $i = n$ then $\underline{\pi_X^{(n,n)}}$ is the identity on $\underline{\Omega^n X}$.

Adding an $\mathrm{Add}(R)$-summand to $\Sigma^{n-i}\Omega^n X$ if necessary, we may take the morphism $\pi_X^{(n,i)}$ as cohomologically surjective. Under such a circumstance, we make a triangle

$$\Delta^{(n,i)}(X) \longrightarrow \Sigma^{n-i}\Omega^n X \xrightarrow{\pi_X^{(n,i)}} \Omega^i X \longrightarrow \Delta^{(n,i)}(X)[1]$$

and define $\Delta^{(n,i)}(X) \in \mathscr{K}(R)$ by this triangle.

Note that there is a short exact sequence of graded R-modules;

$$0 \longrightarrow H(\Delta^{(n,i)}(X)) \longrightarrow H(\Sigma^{n-i}\Omega^n X) \xrightarrow{H(\pi_X^{(n,i)})} H(\Omega^i X) \longrightarrow 0,$$

for all $X \in \mathscr{K}(R)$ and $0 \le i \le n$.

Since $\underline{\pi_X^{(n,i)}}$ is uniquely determined as a morphism in $\underline{\mathscr{K}(R)}$, Theorem 6.6 implies the following lemma.

Lemma 10.1. *For each $X \in \mathscr{K}(R)$ and positive integers $0 \le i \le n$, the complex $\Delta^{(n,i)}(X)$ defined above is uniquely determined as an object of $\underline{\mathscr{K}(R)}$.*

As in the previous section we have triangles of the form;

$$\Omega^{i+1}X \xrightarrow{q_{i+1}} F_i \xrightarrow{p_i} \Omega^i X \longrightarrow \Omega^{i+1}X[1],$$

$$\Sigma^i\Omega^n X \xrightarrow{q^{n-i}} G_{n-i-1} \xrightarrow{p^{n-i-1}} \Sigma^{i+1}\Omega^n X \longrightarrow \Sigma^i\Omega^n X[1],$$

where F_i, $G_{n-i-1} \in \mathrm{Add}(R)$ and p_i (resp. q^{n-i}) is a right (resp. left) $\mathrm{Add}(R)$-approximation for all $0 \le i < n$.

Thus, setting as v_n the identity morphism on $\Omega^n X$, by the induction on $n-i$, we find morphisms $v_i : \Sigma^{n-i}\Omega^n X \to \Omega^i X$ and $a_i : G_i \to F_i$ that make the following diagrams commutative:

$$\begin{array}{ccccccc}
\Omega^{i+1}X & \xrightarrow{q_{i+1}} & F_i & \xrightarrow{p_i} & \Omega^i X & \longrightarrow & \Omega^{i+1}X[1] \\
\uparrow{\scriptstyle v_{i+1}} & & \uparrow{\scriptstyle a_i} & & \uparrow{\scriptstyle v_i} & & \uparrow{\scriptstyle v_{i+1}[1]} \\
\Sigma^{n-i-1}\Omega^n X & \xrightarrow{q^{i+1}} & G_i & \xrightarrow{p^i} & \Sigma^{n-i}\Omega^n X & \longrightarrow & \Sigma^{n-i-1}\Omega^n X[1],
\end{array}$$

for $0 \le i < n$. Here we can take such a_i to be surjective graded R-module homomorphisms. (Actually, if necessary, we may add F_i to G_i, and $q_{i+1}v_{i+1}$ to q^{i+1} as a direct summand.) Then, since $p_i a_i$ is cohomologically surjective, we see that v_i is also cohomologically surjective. Therefore we may take all such v_i to be equal to $\pi_X^{(n,i)}$ for $0 \le i < n$.

Thus we have a commutative diagram in which the rows are $\mathscr{K}(R)$-exact sequences;

$$(10.2)\quad \begin{array}{ccccccccccccc}
0 & \longrightarrow & \Omega^n X & \xrightarrow{q_n} & F_{n-1} & \xrightarrow{f_{n-1}} & \cdots & \xrightarrow{f_1} & F_0 & \xrightarrow{p_0} & X & \longrightarrow & 0 \\
 & & \| & & \uparrow{\scriptstyle a_{n-1}} & & & & \uparrow{\scriptstyle a_0} & & \uparrow{\scriptstyle \pi_X^{(n,0)}} & & \\
0 & \longrightarrow & \Omega^n X & \xrightarrow{q^n} & G_{n-1} & \xrightarrow{g^{n-1}} & \cdots & \xrightarrow{g^1} & G_0 & \xrightarrow{p^0} & \Sigma^n\Omega^n X & \longrightarrow & 0,
\end{array}$$

where $F_i \in \mathrm{Add}(R)$, $f_i = q_i p_i$ and $g^i = q^i p^i$ for $1 \le i < n$. This diagram is divided into two part of commutative diagrams whose rows are $\mathscr{K}(R)$-exact sequences as well:

(10.3)

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & \Omega^i X & \xrightarrow{q_i} & F_{i-1} & \xrightarrow{f_{i-1}} & \cdots & \xrightarrow{f_1} & F_0 & \xrightarrow{p_0} & X & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \pi_X^{(n,i)}} & & \uparrow{\scriptstyle a_{i-1}} & & & & \uparrow{\scriptstyle a_0} & & \uparrow{\scriptstyle \pi_X^{(n,0)}} & & \\
0 & \longrightarrow & \Sigma^{n-i}\Omega^n X & \xrightarrow{q^i} & G_{i-1} & \xrightarrow{g^{i-1}} & \cdots & \xrightarrow{g^1} & G_0 & \xrightarrow{p^0} & \Sigma^n\Omega^n X & \longrightarrow & 0,
\end{array}$$

(10.4)

$$\begin{CD}
0 @>>> \Omega^n X @>{q_n}>> F_{n-1} @>{f_{n-1}}>> \cdots @>{f_{i+1}}>> F_i @>{p_i}>> \Omega^i X @>>> 0 \\
@. @| @A{a_{i-1}}AA @. @A{a_i}AA @A{\pi_X^{(n,i)}}AA @. \\
0 @>>> \Omega^n X @>{q^n}>> G_{n-1} @>{g^{n-1}}>> \cdots @>{g^{i+1}}>> G_i @>{p^i}>> \Sigma^{n-i}\Omega^n X @>>> 0,
\end{CD}$$

Now set $L_i = \mathrm{Ker}\ a_i$ the kernel as a graded R-module homomorphism for $0 \leq i \leq n$. Since each a_i is surjective as a graded R-module homomorphism, we see that $L_i \in \mathrm{Add}(R)$. Then the successive use of octahedron axiom to the diagram (10.4) will show that there is a commutative diagram whose columns are triangles and rows are $\mathscr{K}(R)$-exact sequences:

(10.5)

$$\begin{CD}
0 @>>> \Omega^n X @>{q_n}>> F_{n-1} @>{f_{n-1}}>> \cdots @>{f_{i+1}}>> F_i @>{p_i}>> \Omega^i X @>>> 0 \\
@. @| @A{a_{n-1}}AA @. @A{a_i}AA @A{\pi_X^{(n,i)}}AA @. \\
0 @>>> \Omega^n X @>{q^n}>> G_{n-1} @>{g^{n-1}}>> \cdots @>{g^1}>> G_i @>{p^i}>> \Sigma^{n-i}\Omega^n X @>>> 0 \\
@. @. @A{b^{n-1}}AA @. @A{b^i}AA @AAA @. \\
@. 0 @>>> L_{n-1} @>{\ell^{n-1}}>> \cdots @>{\ell^{i+1}}>> L_i @>>> \Delta^{(n,i)}(X) @>>> 0
\end{CD}$$

In fact we prove by induction on $n-i$ that the third row of the diagram (10.5) is a $\mathscr{K}(R)$-exact sequence. If $n-i=1$ then the following octahedron diagram proves this.

$$\begin{CD}
@. L_{n-1} @= \Delta^{(n,n-1)}(X) @. \\
@. @AAA @AAA @. \\
\Omega^n X @>{q_n}>> F_{n-1} @>{p_{n-1}}>> \Omega^{n-1} X @>>> \Omega^n X \\
@| @A{a_{n-1}}AA @A{\pi_X^{(n,n-1)}}AA @| \\
\Omega^n X @>{q^n}>> G_{n-1} @>{p^{n-1}}>> \Sigma\Omega^n X @>>> \Omega X \\
@. @A{b^{n-1}}AA @AAA @. \\
@. L_{n-1} @= \Delta^{(n,n-1)}(X) @.
\end{CD}$$

If $n-i \geq 2$, then applying the induction hypothesis, we see that in the diagram

$$
\begin{array}{ccccccccccccc}
0 & \longrightarrow & \Omega^n X & \xrightarrow{q_n} & F_{n-1} & \xrightarrow{f_{n-1}} & \cdots & \xrightarrow{f_{i+2}} & F_{i+1} & \xrightarrow{p_{i+1}} & \Omega^{i+1}X & \longrightarrow & 0 \\
 & & \Vert & & \uparrow{\scriptstyle a_{n-1}} & & & & \uparrow{\scriptstyle a_{i+1}} & & \uparrow{\scriptstyle \pi_X^{(n,i+1)}} & & \\
0 & \longrightarrow & \Omega^n X & \xrightarrow{q^n} & G_{n-1} & \xrightarrow{g^{n-1}} & \cdots & \xrightarrow{g^{i+2}} & G_{i+1} & \xrightarrow{p^{i+1}} & \Sigma^{n-i-1}\Omega^n X & \longrightarrow & 0 \\
 & & & & \uparrow{\scriptstyle b^{n-1}} & & & & \uparrow{\scriptstyle b^{i+1}} & & \uparrow & & \\
 & & 0 & \longrightarrow & L_{n-1} & \xrightarrow{\ell^{n-1}} & \cdots & \xrightarrow{\ell^{i+2}} & L_{i+1} & \longrightarrow & \Delta^{(n,i+1)}(X) & \longrightarrow & 0,
\end{array}
$$

the third row is $\mathscr{K}(R)$-exact. On the other hand, by virtue of the so-called 9 lemma, there is a commutative diagram where all rows and columns are triangles:

$$
\begin{array}{ccccc}
\Omega^{i+1}X & \xrightarrow{q_{i+1}} & F_i & \xrightarrow{p_i} & \Omega^i X \\
\uparrow{\scriptstyle \pi_X^{(n,i+1)}} & & \uparrow{\scriptstyle a_i} & & \uparrow{\scriptstyle \pi_X^{(n,i)}} \\
\Sigma^{n-i-1}\Omega^n X & \xrightarrow{q^{i+1}} & G_i & \xrightarrow{p^i} & \Sigma^{n-i}\Omega^n X \\
\uparrow & & \uparrow{\scriptstyle b_i} & & \uparrow \\
\Delta^{(n,i+1)}(X) & \longrightarrow & L_i & \longrightarrow & \Delta^{(n,i)}(X)
\end{array}
$$

In particular we have a $\mathscr{K}(R)$-exact sequence

$$
0 \longrightarrow \Delta^{(n,i+1)}(X) \longrightarrow L_i \longrightarrow \Delta^{(n,i)}(X) \longrightarrow 0
$$

Combining the sequences above we finally obtain the $\mathscr{K}(R)$-exact sequence:

$$
0 \longrightarrow L_{n-1} \xrightarrow{\ell^{n-1}} L_{n-2} \longrightarrow \cdots \xrightarrow{\ell^{i+1}} L_i \longrightarrow \Delta^{(n,i)}(X) \longrightarrow 0 \tag{10.6}
$$

This proves that all the rows in the diagram (10.5) are $\mathscr{K}(R)$-exact sequences.

Letting $\widetilde{L^{(n,i)}}$ be the contraction of the Add(R)-resolution (10.6), we have the isomorphism $\Delta^{(n,i)}(X) \cong \widetilde{L^{(n,i)}}$. We have thus proved the following theorem.

Theorem 10.2. *Let $X \in \mathscr{K}(R)$ and $0 \leq i \leq n$. Then $\Delta^{(n,i)}(X)$ has a finite* Add(R)*-resolution of length $n-i-1$. In this case $\Delta^{(n,i)}(X)$ is isomorphic in $\mathscr{K}(R)$ to the contraction of such a finite* Add(R)*-resolution.*

Remark 10.3.

(1) If R is a Gorenstein ring of dimension zero (i.e. a self-injective algebra), then we can take all the a_i are isomorphisms and hence one can take $L_i = 0$ for all $0 \leq i < n$. Thus we have $\underline{\Delta^{(n,i)}(X)} = 0$ or $\Delta^{(n,i)}(X) \in \mathrm{Add}(R)$ for all $0 \leq i \leq n$. Moreover for any choice of a_i the sequence (10.6) is a split sequence in this case.

(2) Let S be a multiplicatively closed subset of R. Then it is clear that the construction of the diagram (10.6) is commutative with taking localization by S.

As a consequence of this, we observe an isomorphism

$$S^{-1}\Delta_R^{(n,i)}(X) \cong \Delta_{S^{-1}R}^{(n,i)}(S^{-1}X),$$

in the stable category $\underline{\mathscr{K}(S^{-1}R)}$ for all $0 \le i < n$. Moreover the localized sequence of (10.6) by S is an $\mathrm{Add}(S^{-1}R)$-resolution of $\Delta_{S^{-1}R}^{(n,i)}(S^{-1}X)$.

By this remark, if R is a generically Gorenstein ring, then the $\mathrm{Add}(R)$-resolution (10.6) is generically split. Thus we have proved the following theorem.

Theorem 10.4. *Let R be a generically Gorenstein ring. For any $X \in \mathscr{K}(R)$ and $0 \le i \le n$, $\Delta^{(n,i)}(X)$ has a finite $\mathrm{Add}(R)$-resolution of length $n-i-1$ that is generically split.*

11. The main theorem and the proof

The following theorem is one of the most essential observations to prove the main theorem.

Theorem 11.1. *Let R be a generically Gorenstein ring, and let $X \in \mathscr{K}(R)$. If $H(X^*) = 0$, then $\Omega^r X$ is *torsion-free for each non-negative integer r.*

To prove the theorem we prepare several lemmas.

Lemma 11.2. *Let X be a complex in $\mathscr{K}(R)$ and assume that $H(X^*) = 0$. Then we have $\mathrm{Hom}_{\mathscr{K}(R)}(X, F) = 0$ for all $F \in \mathrm{Add}(R)$.*

Proof. Note that $H(X^*)$ is the cohomology module of the complex $\mathrm{Hom}_R(X, R)$, hence we have the equality $H(X^*) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{K}(R)}(X, R[i])$. Thus if $H(X^*) = 0$, then we see that $\mathrm{Hom}_{\mathscr{K}(R)}(X, P[i]) = 0$ for any finitely generated projective R-module P and an integer i. Recall from Theorem 5.8 and Proposition 5.7 that any complex $F \in \mathrm{Add}(R)$ is isomorphic to a direct sum $\bigoplus_{i\in\mathbb{Z}} F^i[-i]$ with F^i being a projective R-module for each $i \in \mathbb{Z}$. On the other hand it follows from Lemma 5.1 the direct sum is a product in $\mathscr{K}(R)$. Hence $\mathrm{Hom}_{\mathscr{K}(R)}(X, F) = \prod_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{K}(R)}(X, F^i[-i]) = 0$ as desired. □

Lemma 11.3. *Let $X, Y \in \mathscr{K}(R)$. Assume the following conditions:*

(1) *Y has an $\mathrm{Add}(R)$-resolution of finite length.*

(2) *$H(X^*) = 0$.*

Then we have $\mathrm{Hom}_{\mathscr{K}(R)}(X, Y) = 0$.

Proof. This is obvious from the previous lemma and utilizing the induction on the length ℓ of the $\mathrm{Add}(R)$-resolution of Y. In fact, if $\ell = 0$ then $Y \in \mathrm{Add}(R)$ hence $\mathrm{Hom}_{\mathscr{K}(R)}(X, Y) = 0$ by Lemma 11.2. If $\ell > 0$ then there is a triangle

$$Y' \longrightarrow F_0 \longrightarrow Y \longrightarrow Y'[1]\ ,$$

where $F_0 \in \mathrm{Add}(R)$ and Y' has an $\mathrm{Add}(R)$-resolution of length $\ell - 1$. Thus $\mathrm{Hom}_{\mathscr{K}(R)}(X, Y'[i]) = 0$ for all $i \in \mathbb{Z}$ by the induction hypothesis. Since there is an exact sequence of R-modules;

$$\mathrm{Hom}_{\mathscr{K}(R)}(X, F_0) \longrightarrow \mathrm{Hom}_{\mathscr{K}(R)}(X, Y) \longrightarrow \mathrm{Hom}_{\mathscr{K}(R)}(X, Y'[1]),$$

which results that $\mathrm{Hom}_{\mathscr{K}(R)}(X, Y) = 0$. $\square$

Now we proceed to the proof of Theorem 11.1.

In the following of this proof we assume that R is generically Gorenstein and $H(X^*) = 0$. It is clear that X is *torsion-free, since $H(X^*) = 0 \to H(X)^*$ is injective. We shall prove that so is $\Omega^r X$ for $r \geq 1$.

Let $n \geq 2$ be an integer. We have the following commutative diagram from (10.5);

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & \Omega^n X & \xrightarrow{q_n} & F_{n-1} & \xrightarrow{f_{n-1}} & \cdots & \xrightarrow{f_1} & F_i & \xrightarrow{p_0} & X & \longrightarrow & 0 \\
 & & \| & & \uparrow{\scriptstyle a_{n-1}} & & & & \uparrow{\scriptstyle a_0} & & \uparrow{\scriptstyle \pi_X^{(n,0)}} & & \\
0 & \longrightarrow & \Omega^n X & \xrightarrow{q^n} & G_{n-1} & \xrightarrow{g^{n-1}} & \cdots & \xrightarrow{g^1} & G_i & \xrightarrow{p^0} & \Sigma^n \Omega^n X & \longrightarrow & 0 \\
 & & & & \uparrow{\scriptstyle b^{n-1}} & & & & \uparrow{\scriptstyle b^0} & & \uparrow & & \\
 & & 0 & \longrightarrow & L_{n-1} & \xrightarrow{\ell^{n-1}} & \cdots & \xrightarrow{\ell^1} & L_0 & \longrightarrow & \Delta^{(n,0)}(X) & \longrightarrow & 0,
\end{array} \tag{11.1}$$

where the rows are $\mathscr{K}(R)$-exact sequences and the columns are triangles. Taking the contracted triangles of the rows we obtain the following commutative diagram whose rows and columns are triangles:

$$\begin{array}{ccccccc}
 & & \widetilde{L}[1] & \xrightarrow{\cong} & \Delta^{(n,0)}(X)[1] & & \\
 & & \uparrow{\scriptstyle \lambda} & & \uparrow{\scriptstyle \sigma} & & \\
\Omega^n X[n-1] & \xrightarrow{\psi_n^F} & \widetilde{F} & \xrightarrow{\varphi_n^F} & X & \xrightarrow{\widetilde{\omega_n}^F} & \Omega^n X[n] \\
\| & & \uparrow{\scriptstyle \widetilde{a}} & & \uparrow{\scriptstyle \pi_X^{(n,0)}} & & \| \\
\Omega^n X[n-1] & \xrightarrow{\psi_n^G} & \widetilde{G} & \xrightarrow{\varphi_n^G} & \Sigma^n \Omega^n X & \xrightarrow{\widetilde{\omega_n}^G} & \Omega^n X[n] \\
 & & \uparrow{\scriptstyle \widetilde{b}} & & \uparrow{\scriptstyle \tau} & & \\
 & & \widetilde{L} & \xrightarrow{\cong} & \Delta^{(n,0)}(X), & &
\end{array} \tag{11.2}$$

where $\widetilde{a}$ and $\widetilde{b}$ are the induced morphism from $\{a_i\}$ and $\{b_i\}$ respectively. See Definition 8.4 and Theorem 8.5. (Since the third column is a triangle, one can see that the second column is also a triangle. But this fact will be also easily seen by the construction of $\Delta^{(n,i)}(X)$ in the previous section.) We know from Theorem 10.2 that $\Delta^{(n,0)}(X)[1]$ has a finite $\mathrm{Add}(R)$-resolution of finite length. Hence it follows from Lemma 11.3 we have that σ in the diagram is zero. Thus λ is also zero in the diagram by the commutativity of the upper square. This means that the second and the third columns are split triangles, hence $\widetilde{a}$ and $\pi_X^{(n,0)}$ have right inverses. Notice from this that $\widetilde{L}$, and hence $\Delta^{(n,0)}(X)$ as well, is *torsion-free, since it is a direct summand of $\Sigma^n\Omega^n X$.

Note that the following diagram is commutative (cf. Theorem 8.5).

(11.3)
$$\begin{array}{ccc}
\widetilde{F} & \xrightarrow{pr^F_{n-1}} & F_{n-1}[n-1] \\
\Big\uparrow{\scriptstyle \widetilde{a}} & & \Big\uparrow{\scriptstyle a_{n-1}[n-1]} \\
\widetilde{G} & \xrightarrow{pr^G_{n-1}} & G_{n-1}[n-1] \\
\Big\uparrow{\scriptstyle \widetilde{b}} & & \Big\uparrow{\scriptstyle b_{n-1}[n-1]} \\
\widetilde{L} & \xrightarrow{pr^L_{n-1}} & L_{n-1}[n-1]
\end{array}$$

We shall now prove that $pr^L_{n-1} = 0$ in $\mathscr{K}(R)$.

To prove this we note that $\widetilde{L}$ has an $\mathrm{Add}(R)$-resolution of the form (10.6) that is generically split. Therefore we see from Corollary 9.3 that $S^{-1}(pr^L_{n-1}) = 0$ in $\mathcal{K}(S^{-1}R)$, where S is the set of all non-zero divisors in R as before. Since $\widetilde{L}$ is *torsion-free and $L_{n-1} \in \mathrm{Add}(R)$, it follows from Theorem 5.10 that $pr^L_{n-1} = 0$ in $\mathscr{K}(R)$ as desired.

Then we have $pr^G_{n-1}\widetilde{b} = 0$ by the commutativity of the diagram (11.3). Hence there is a morphism $e : \widetilde{F} \to G_{n-1}[n-1]$ in $\mathscr{K}(R)$ such that $e\,\widetilde{a} = pr^G_{n-1}$.

Let $\widetilde{F'}$ and $\widetilde{G'}$ be the contractions of the partial $\mathrm{Add}(R)$-resolutions appeared in the following diagram, thus $\widetilde{F'} = F_{n-2}[n-2] \oplus \cdots \oplus F_0 \subseteq \widetilde{F} = F_{n-1}[n-1] \oplus F_{n-2}[n-2] \oplus \cdots \oplus F_0$ and the same for $\widetilde{G'}$.

(11.4)
$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & \Omega^{n-1}X & \xrightarrow{q_{n-1}} & F_{n-2} & \xrightarrow{f_{n-2}} & \cdots & \xrightarrow{f_1} & F_0 & \xrightarrow{p_0} & X & \longrightarrow & 0 \\
 & & \Big\uparrow{\scriptstyle \pi_X^{(n,n-1)}} & & \Big\uparrow{\scriptstyle a_{n-2}} & & & & \Big\uparrow{\scriptstyle a_0} & & \Big\uparrow{\scriptstyle \pi_X^{(n,0)}} & & \\
0 & \longrightarrow & \Sigma\Omega^n X & \xrightarrow{q^{n-1}} & G_{n-2} & \xrightarrow{g^{n-2}} & \cdots & \xrightarrow{g^1} & G_0 & \xrightarrow{p^0} & \Sigma^n\Omega^n X & \longrightarrow & 0
\end{array}$$

We notice that $\widetilde{F'}$ and $\widetilde{G'}$ are subcomplexes of $\widetilde{F}$ and $\widetilde{G}$ respectively. Recall that $\widetilde{a}$ is a splitting epimorphism in $\mathscr{K}(R)$ and it is represented by a lower triangle matrix whose diagonal entries are $a_{n-1}, \cdots, a_0$ which are all split epimorphisms of graded R-modules. Thus we have that $\widetilde{F'} = \widetilde{a}(\widetilde{G'})$.

There is a diagram whose rows are triangles and squares are commutative;

(11.5)
$$\begin{array}{ccccc} \widetilde{F'} & \longrightarrow & \widetilde{F} & \xrightarrow{pr^F_{n-1}} & F_{n-1}[n-1] \\ \uparrow{\scriptstyle \widetilde{a'}} & & \uparrow{\scriptstyle \widetilde{a}} & \searrow{\scriptstyle e} & \uparrow{\scriptstyle a_{n-1}[n-1]} \\ \widetilde{G'} & \longrightarrow & \widetilde{G} & \xrightarrow{pr^G_{n-1}} & G_{n-1}[n-1] \end{array}$$

Since $pr^G_{n-1}(\widetilde{G'}) = 0$, we have $e(\widetilde{F'}) = e\widetilde{a}(\widetilde{G'}) = pr^G_{n-1}(\widetilde{G'}) = 0$. Thus e induces a morphism $f : F_{n-1}[n-1] \to G_{n-1}[n-1]$ such that $e = f\ pr^F_{n-1}$. Hence it holds that $f\ pr^F_{n-1}\widetilde{a} = pr^G_{n-1}$.

Now recalling in the diagram (11.1) that $q_n[n-1] = pr^F_{n-1}\psi^F_n$ and $q^n[n-1] = pr^G_{n-1}\psi^G_n$ by Theorem and Definition 8.2, we have equalities;

$$q^n[n-1] = pr^G_{n-1}\psi^G_n = f\ pr^F_{n-1}\widetilde{a}\ \psi^G_n = f\ pr^F_{n-1}\psi^F_n = f\ q_n[n-1].$$

This shows the commutativity of the following diagram in which the rows are triangles:

$$\begin{array}{ccccc} \Omega^n X[n-1] & \xrightarrow{q_n[n-1]} & F_{n-1}[n-1] & \longrightarrow & \Omega^{n-1}X[n-1] \\ \| & & \downarrow{\scriptstyle f} & & \\ \Omega^n X[n-1] & \xrightarrow{q^n[n-1]} & G_{n-1}[n-1] & \longrightarrow & \Sigma\Omega^n X[n-1] \end{array}$$

Recall that $q^n[n-1]$ is a left $\mathrm{Add}(R)$-approximation. Then it is easy to see from Remark 7.6 that $q_n[n-1]$ is also a left $\mathrm{Add}(R)$-approximation. As a consequence of this we have $\Omega^{n-1}X[n-1] \cong \Sigma\Omega^n X[n-1]$ in $\underline{\mathscr{K}(R)}$. Thus $\Omega^{n-1}X$ is *torsion-free by Theorem 7.14. Since n is any integer not less than 2, this completes the proof of Theorem 11.1. □

Theorem 11.1 can be strengthened as in the following form.

Theorem 11.4. *Let R be a generically Gorenstein ring. Assume $H(X^*) = 0$ for $X \in \mathscr{K}(R)$. Then $\Omega^r X$ is *reflexive for any non-negative integer r.*

Proof. Let r be a non-negative integer. Note from the definition that there is a triangle

$$\Omega^{r+1}X \xrightarrow{q} F_r \xrightarrow{p} \Omega^r X \longrightarrow \Omega^{r+1}X[1],$$

where p is a right $\mathrm{Add}(R)$-approximation. Hence the sequence of graded R-modules;

$$0 \longrightarrow H(\Omega^{r+1}X) \xrightarrow{H(q)} H(F_r) \xrightarrow{H(p)} H(\Omega^r X) \longrightarrow 0$$

is exact. Given a graded R-module homomorphism $\alpha : H(\Omega^r X) \to R[i]$ for some $i \in \mathbb{Z}$, we find a morphism $b : F_r \to R[i]$ in $\mathscr{K}(R)$ with $H(b) = \alpha H(p)$, since F_r is *reflexive. Then we have $H(bq) = H(b)H(q) = \alpha H(pq) = 0$. As we have shown in Theorem 11.1, $\Omega^{r+1}X$ is *torsion-free, we have that $bq = 0$. Then by a property of exact triangles there is an $a : \Omega^r X \to R[i]$ that satisfies $b = ap$. Since $H(p)$ is a surjection, it thus follows that $H(a) = \alpha$. Then one can apply Lemma 3.2(2) to conclude that $\Omega^r X$ is *reflexive. □

Proposition 11.5. *Let $Y \in \mathscr{K}(R)$. Assume the following conditions are satisfied:*

(1) *Y is *torsion-free.*

(2) *ΩY is *reflexive.*

Then we have $\mathrm{Ext}^1_R(H(Y), R) = 0$.

Proof. From the definition of ΩY there is a triangle in $\mathscr{K}(R)$;

$$\Omega Y \xrightarrow{q} F \xrightarrow{p} Y \xrightarrow{\omega} \Omega Y[1],$$

where p is a right $\mathrm{Add}(R)$-approximation. Hence there is an exact sequence of graded R-modules;

$$0 \longrightarrow H(\Omega Y) \xrightarrow{H(q)} H(F) \xrightarrow{H(p)} H(Y) \longrightarrow 0,$$

where $H(F)$ is a projective graded R-module. Thus we have an exact sequence

$$0 \longrightarrow H(Y)^* \xrightarrow{H(p)^*} H(F)^* \xrightarrow{H(q)^*} H(\Omega Y)^* \longrightarrow \mathrm{Ext}^1_R(H(Y), R) \longrightarrow 0.$$

On the other hand we also have a triangle;

$$Y^* \xrightarrow{p^*} F^* \xrightarrow{q^*} (\Omega Y)^* \xrightarrow{\omega^*[1]} Y^*[1].$$

Therefore we have the following commutative diagram of R-modules whose rows are exact sequences of graded R-modules:

$$\begin{array}{ccccccccc}
H((\Omega Y)^*)[-1] & \xrightarrow{H(\omega^*)} & H(Y^*) & \xrightarrow{H(p^*)} & H(F^*) & \xrightarrow{H(q^*)} & H((\Omega Y)^*) & \xrightarrow{H(\omega^*[1])} & H(Y^*[1]) \\
 & & \downarrow{\scriptstyle \rho_{Y,R}} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \rho_{\Omega Y,R}} & & \\
0 & \longrightarrow & H(Y)^* & \xrightarrow{H(p)^*} & H(F)^* & \xrightarrow{H(q)^*} & H(\Omega Y)^* & \longrightarrow & \mathrm{Ext}^1_R(H(Y), R) \to 0.
\end{array}$$

Since Y is *torsion-free, $\rho_{Y,R}$ is injective and hence so is $H(p^*)$. It thus follows that $H(\omega^*) = 0$. Then we must have that $H(q^*)$ is surjective. Since ΩY is *reflexive, $\rho_{\Omega Y,R}$ is bijective. As a result we have that $H(q)^*$ is surjective as well as $H(q^*)$. Thus it is concluded from the exactness of the second row that $\mathrm{Ext}^1_R(H(Y), R) = 0$. □

Combining Proposition 11.4 with Proposition 11.5, we obtain the following proposition that is a key for the proof of Theorem 11.7.

Proposition 11.6. *Let R be a generically Gorenstein ring, and assume that $H(X^*) = 0$ for $X \in \mathscr{K}(R)$. Then we have*

$$\mathrm{Ext}^r_R(H(X), R) = 0,$$

for all $r > 0$.

Proof. Recall from the argument after Theorem 9.7 that one can take a partial Add(R)-resolution of X

$$0 \longrightarrow \Omega^r X \xrightarrow{q_r} F_{r-1} \xrightarrow{f_{r-1}} \cdots \longrightarrow F_1 \xrightarrow{f_1} F_0 \xrightarrow{p_0} X \longrightarrow 0,$$

such that it induces an exact sequence of graded R-modules

$$0 \longrightarrow H(\Omega^r X) \xrightarrow{H(q_r)} H(F_{r-1}) \xrightarrow{H(f_{r-1})} \cdots \longrightarrow H(F_1) \xrightarrow{H(f_1)} H(F_0) \xrightarrow{H(p_0)} H(X) \longrightarrow 0.$$

Since R is generically Gorenstein and $H(X^*) = 0$, it follows from Theorem 11.4 that $\Omega^r X$ is *reflexive for each $r > 0$. Note also that X is *torsion-free. Then, by Proposition 11.5, we have $\mathrm{Ext}^1_R(H(\Omega^{r-1}X), R) = 0$ for all $r > 0$. Thus it follows from the long exact sequence above of graded R-modules that $\mathrm{Ext}^r_R(H(X), R) = 0$ for $r > 0$. □

The following is the main theorem of this paper, which we can now prove as a result of the previous theorems and propositions.

Theorem 11.7. *Let R be a generically Gorenstein ring, and let $X \in \mathscr{K}(R)$. Then, $H(X) = 0$ if and only if $H(X^*) = 0$.*

Proof. We have only to prove that if $H(X^*) = 0$ then $H(X) = 0$ under the assumption that R is generically Gorenstein. The other implication follows from this by the duality $X^{**} \cong X$. Thus in this proof we assume that $H(X^*) = 0$ and our aim is to show that $H(X) = 0$.

(1st step): We may assume that $(R, \mathfrak{m}, k)$ is a local ring, which is generically Gorenstein. Furthermore we may assume that $\dim R > 0$.

Note that $H(X) = 0$ if and only if $H(X_\mathfrak{m}) = H(X)_\mathfrak{m} = 0$ for all maximal ideal $\mathfrak{m}$ of R, and that $\mathrm{Hom}_R(X, R)_\mathfrak{m} = \mathrm{Hom}_{R_\mathfrak{m}}(X_\mathfrak{m}, R_\mathfrak{m})$. It is also obvious that if R is generically Gorenstein, then so are all $R_\mathfrak{m}$. The first half is clear from these observations.

If $\dim R = 0$ then R is a Gorenstein ring by the generic Gorenstein assumption and the theorem is trivial in this case, since R is an injective R-module. Hence we may avoid this case.

(2nd step): We may assume that $\mathfrak{m}H(X) = 0$.

To show this, let $x \in \mathfrak{m}$ and consider the Koszul complex

$$K(x) = \left[\, 0 \longrightarrow R \xrightarrow{x} R \longrightarrow 0 \,\right].$$

Set $X' = X \otimes_R K(x)$, and since there is a triangle $X \xrightarrow{x} X \longrightarrow X' \longrightarrow X[1]$ in $\mathscr{K}(R)$, we have the equivalence $H(X) = 0 \;\Leftrightarrow\; H(X') = 0$ by Nakayama Lemma. Since X'^* is of the same form as X', we can see that $H(X^*) = 0 \;\Leftrightarrow\; H(X'^*) = 0$ as well.

Now take a generating set $x_1, \dots, x_m$ of the maximal ideal $\mathfrak{m}$, and consider the Koszul complex $X'' = X \otimes_R K(x_1, \dots, x_m) = X \otimes_R K(x_1) \otimes_R \cdots \otimes_R K(x_m)$. Then we have the equivalences $H(X'') = 0 \;\Leftrightarrow\; H(X) = 0$, and also $H(X''^*) = 0 \;\Leftrightarrow\; H(X^*) = 0$. Thus it is enough to show that $H(X''^*) = 0$ implies $H(X'') = 0$. It is also clear that for any element $x \in \mathfrak{m}$, the multiplication map on X'' is trivial in $\mathscr{K}(R)$, hence $\mathfrak{m}H(X'') = 0$.

(3rd step): Now assume that $H(X) \neq 0$. Then there is an integer i with $H^i(X) \neq 0$. By the second step of this proof, $H^i(X)$ is a non-trivial k-module, where $k = R/\mathfrak{m}$. On the other hand we have shown in Proposition 11.6 that $\mathrm{Ext}^r_R(H(X), R) = 0$ for all $r > 0$ under the condition that $H(X^*) = 0$. Therefore we have

$$\mathrm{Ext}^r_R(k, R) = 0 \quad \text{for all} \quad r > 0.$$

This requires that R is a Gorenstein ring of dimension zero, which is not the case by the first step. Hence $H(X) = 0$ and the proof of the theorem is completed. □

12. Applications

Recall that a chain homomorphism f between complexes is called a quasi-isomorphism if the cohomology mapping $H(f)$ is an isomorphism of modules.

Theorem 12.1 (Corollary 1.2). *Assume that the ring R is a generically Gorenstein ring. Let $f : X \to Y$ be a morphism in $\mathscr{K}(R)$. Then, f is a quasi-isomorphism if and only if the R-dual $f^* : Y^* \to X^*$ is a quasi-isomorphism.*

Proof. In fact, let $X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1]$ be a triangle in $\mathscr{K}(R)$. Then f is a quasi-isomorphism if and only if $H(Z) = 0$. We have shown in Theorem 11.7 that $H(Z) = 0$ if and only if $H(Z^*) = 0$. Since $Z^*[-1] \longrightarrow Y^* \xrightarrow{f^*} X^* \longrightarrow Z^*$ is a triangle, Theorem 12.1 follows. □

Now we recall the definition of totally reflexive modules. A finitely generated module M over a commutative Noetherian ring R is called a totally reflexive module or a module of G-dimension zero if $\mathrm{Ext}^i_R(M, R) = \mathrm{Ext}^i_R(\mathrm{Tr}M, R) = 0$ for all $i > 0$. This is equivalent to the following three conditions;

(i) M is reflexive, i.e. the natural mapping $M \to M^{**}$ is bijective.

(ii) $\mathrm{Ext}^i_R(M, R) = 0$ for all $i > 0$.

(iii) $\mathrm{Ext}^i_R(M^*, R) = 0$ for all $i > 0$.

See [2] for the detail of totally reflexive modules. The following theorem says that only the condition (ii) is sufficient for totally reflexivity if the ring R is generically Gorenstein.

Theorem 12.2 (Corollary 1.3). *Assume that the ring R is a generically Gorenstein ring. Let M be a finitely generated R-module. Then the following conditions are equivalent:*

(1) *M is a totally reflexive R-module.*

(2) $\mathrm{Ext}^i_R(M, R) = 0$ *for all* $i > 0$.

(3) *M is an infinite syzygy, i.e. there is an exact sequence of infinite length of the form* $0 \longrightarrow M \longrightarrow P_0 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow \cdots$, *where each P_i is a finitely generated projective R-module.*

Proof. The implications (1) ⇒ (2) and (1) ⇒ (3) are well-known and easily proved.

(2) ⇒ (1): Take projective resolutions for M and M^* respectively as

$$\cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \longrightarrow 0 \text{ and } \cdots \xrightarrow{g_2} G_1 \xrightarrow{g_1} G_0 \xrightarrow{g_0} M^* \longrightarrow 0.$$

Then we consider the complex

$$X = \left[\cdots \xrightarrow{g_2} G_1 \xrightarrow{g_1} G_0 \xrightarrow{f_0^* g_0} F_0^* \xrightarrow{f_1^*} F_1^* \xrightarrow{f_2^*} \cdots \right],$$

which belongs to $\mathscr{K}(R)$, and acyclic by the condition (2). Hence by Theorem 11.7 the dual X^* is acyclic as well. Since

$$X^* = \left[\cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \longrightarrow G_0^* \xrightarrow{g_1^*} G_1^* \xrightarrow{g_2^*} \cdots \right],$$

is an exact sequence, it follows that $M \cong M^{**}$ and $\mathrm{Ext}_R^i(M^*, R) = 0$ for $i > 0$.

(3) ⇒ (2): As in (3) we assume that there is an exact sequence

$$0 \longrightarrow M \longrightarrow P_0 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow \cdots .$$

Then combining this with the projective resolution $\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ of M, we have an acyclic complex in $\mathscr{K}(R)$

$$Y = \left[\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow P_0 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow \cdots \right].$$

It follows from Theorem 11.7 that Y^* is acyclic again. In particular, the sequence $F_0^* \longrightarrow F_1^* \longrightarrow F_2^* \longrightarrow \cdots$ is exact, and hence $\mathrm{Ext}_R^i(M, R) = 0$ for $i > 0$. □

Recall that a finitely generated module M has the G-dimension at most n, denoted by $\text{G-dim}_R M \leq n$, if there is an exact sequence of the form

$$0 \longrightarrow G_n \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0,$$

where all G_i $(0 \leq i \leq n)$ are totally reflexive.

Theorem 12.3. *Under the assumption that R is a generically Gorenstein ring, we have the equality*

$$\text{G-dim}_R M = \sup\{ i \in \mathbb{Z} \mid \mathrm{Ext}_R^i(M, R) \neq 0 \},$$

for a finitely generated R-module M.

Proof. Setting $n = \sup\{ i \in \mathbb{Z} \mid \mathrm{Ext}_R^i(M, R) \neq 0 \}$, we have only to consider the case $n < +\infty$. In this case it is easy to see that $n \leq \text{G-dim}_R M$. (This is just because

$\mathrm{Ext}_R^i(M,R)=0$ for any $i>\text{G-dim}_R M$.) Now we take part of projective resolution of M and get the nth syzygy module $\Omega_R^n(M)$, that is,

$$0 \longrightarrow \Omega_R^n(M) \longrightarrow P_{n-1} \longrightarrow P_{n-2} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

is an exact sequence of R-modules with each P_i being projective. Then, since it holds that $\mathrm{Ext}_R^i(\Omega_R^n(M),R)=0$ for $i>0$, $\Omega_R^n(M)$ is totally reflexive by Theorem 12.2. Therefore we have $\text{G-dim}_R M \le n$. □

Jorgensen and Şega [8] gave an example of a module over a non-Gorenstein Artinian ring that disproves the implication (2) ⇒ (1) in Corollary 1.3, hence the generic Gorensteinness assumption in the theorem is indispensable.

The following is a commutative version of Tachikawa conjecture, which we obtain as a corollary to Theorem 11.7. It should be noted that it has been proved by Avramov, Buchweitz and Şega [3].

Theorem 12.4 (Corollary 1.5)**.** *Let R be a Cohen-Macaulay ring with canonical module ω. Furthermore assume that R is a generically Gorenstein ring. If $\mathrm{Ext}_R^i(\omega,R)=0$ for all $i>0$, then R is Gorenstein.*

Proof. Assume $\mathrm{Ext}_R^i(\omega,R)=0$ for all $i>0$. It is enough to show that ω is a projective R-module. We see from Theorem 12.2 that ω is a totally reflexive R-module, and hence there is an exact sequence of the form $0 \longrightarrow \omega \longrightarrow P_0 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow \cdots$, where each P_i is a finitely generated projective R-module. Setting $M=\mathrm{Ker}(P_1 \to P_2)$, we note that M is an maximal Cohen-Macaulay module, since there is an exact sequence $0 \longrightarrow M \longrightarrow P_1 \longrightarrow P_2 \longrightarrow \cdots$.

Therefore we have $\mathrm{Ext}_R^1(M,\omega)=0$ by the local duality theorem. It however means that a short exact sequence $0 \longrightarrow \omega \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ splits, and ω is a direct summand of the projective module P_0, and it is projective. □

Theorem 12.5 (Corollary 1.6)**.** *Assume that the ring R is a generically Gorenstein ring. Let X be a complex of finitely generated projective modules. If the both of $H(X)$ and $H(X^*)$ are bounded above, i.e. $X, X^* \in D^-(R)$, then we have the isomorphism in the derived category:*

$$X \cong \mathrm{RHom}_R(\mathrm{RHom}_R(X,R),R).$$

Proof. Let $f : P \to X$ be a K-projective resolution of X, that is, f is a quasi-isomorphism and P is a projective complex such that $\mathrm{Hom}_R(P,-)$ preserves quasi-isomorphisms. Such a K-projective resolution is known to exists for any X. Furthermore, since $X \in D^-(R)$, we can take such P as $P \in \mathscr{K}(R)$. Similarly we can take a K-projective resolution $g : Q \to X^*$ with $Q \in \mathscr{K}(R)$. Then it follows from Theorem 12.1 that the R-dual $f^* : X^* \to P^*$ is a quasi-isomorphism, and hence Q is isomorphic to $\mathrm{RHom}_R(X,R)$ in the derived category $D(R)$. Note that $f^*g : Q \to P^*$ is a quasi-isomorphism as well. Then again by Theorem 12.1 we see that $g^*f : P \to Q^*$ is also a quasi-isomorphism. This means that X is isomorphic to $\mathrm{RHom}_R(Q,R)$ in $D(R)$. □

As a miscellaneous result we obtain the following.

Theorem 12.6 (Corollary 1.7)**.** *Assume that the ring R is a generically Gorenstein ring. Let X be a complex of finitely generated projective modules.*

If all the cohomology modules $H^i(X)$ $(i \in \mathbb{Z})$ have dimension at most ℓ as R-modules, then so are the modules $H^i(X^*)$ $(i \in \mathbb{Z})$.

Proof. The assumption exactly means that $X_{\mathfrak{p}}$ is acyclic for a prime ideal $\mathfrak{p}$ with $\dim R/\mathfrak{p} > \ell$. Note that each localization $R_{\mathfrak{p}}$ is generically Gorenstein. Therefore $(X^*)_{\mathfrak{p}} = \mathrm{Hom}_{R_{\mathfrak{p}}}(X_{\mathfrak{p}}, R_{\mathfrak{p}})$ is acyclic again for such $\mathfrak{p}$ with $\dim R/\mathfrak{p} > \ell$, by Theorem 11.7. □

Now we introduce the dimension of a complex X as

$$\dim_R X = \sup\{\dim H^i(X) \mid i \in \mathbb{Z}\},$$

which is the dimension of the big support of X in the derived category. (Note that we use the convention that $\dim_R M = -1$ for the trivial R-module $M = \{0\}$.) Then the theorem above includes the following generalization of Theorem 11.7.

Corollary 12.7. *Let R be a generically Gorenstein ring. Then, for a complex $X \in \mathscr{K}(R)$, we have the equality* $\dim_R X = \dim_R X^*$.

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